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TOWARD A CALCULUS OF CONCEPTS

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1. **Introduction.** By *concepts* will be meant propositions (or truth-values), attributes (or classes), and relations of all degrees. The *degree* of a concept will be said to be 0, 1, or n (>1), and the concept will be said to be *medadic*,[2] *monadic*, or *n-adic*, according as the concept is a proposition, an attribute, or an n-adic relation. The common procedure in systematizing logistic is to treat these successive degrees as ultimately separate categories. The partition is not rested upon properties of the thus classified elements within the formal system, but is imposed rather at the metamathematical level, through stipulations as to what combinations of signs are to be accorded or denied meaning. Each function of the formal system is restricted, thus metamathematically, to one degree for its values and to one for each of its arguments. The theory of types imports a further scheme of infinite partition, imposed by metamathematical stipulations as to the relative types of admissible arguments of the several functions and stipulations as to the types of the values of the functions relative to the types of the arguments.

The elaborateness of the metamathematical grillwork which thus underlies formal logistic accounts in part for the tendency of those interested in logistic less for the matters treated than for the structures exemplified to limit their attention to the propositional calculus and the Boolean calculus of attributes (or classes), which, taken separately, are independent of the partitioning. A second reason for the algebraic appeal of these departments is their freedom from bound (apparent) variables: for use of bound variables fuses systematic considerations with notational or metamathematical ones in a way which resists ordinary formulation in terms of fixed functions and their arguments. Freedom from bound variables may be regarded, indeed, as the feature distinguishing algebra from analysis.

The viewpoint is adopted in this paper that it is desirable to defer the described metamathematical complications to the utmost, i.e., to systematize as much as possible of logistic before departing from the simple conception of system illustrated by the propositional and attributary calculi. One aim of this procedure is an analysis of the very difficulties which are thus deferred: for the procedure both isolates those matters and provides a maximum body of antecedent theory on which to rest them. Specifically, it is hoped that those supplementary portions of theory which import types and bound variables can be built from our basic metamathematically simple calculus by means purely of certain quasi definitional notational conventions, roughly along the lines sketched in §12.

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[2] 0-adic. See ***Collected papers of C. S. Peirce***, vol. 3, p. 294.

The immediate concern of this paper is not the developments thus projected, but the basic calculus itself. Primitives and definitions will be presented for a calculus which, like the propositional and attributary calculi, neither involves bound variables nor depends upon informal partition of its elements into types or other categories, but which so exceeds the scope of either of the latter calculi as to embrace both of them and progressively more complex calculi of relations of all degrees as well. The elements of the calculus, represented by the variables 'α', 'β', etc., are concepts. They include no concepts of the higher types—attributes and relations of concepts—, but embrace concepts of lowest type in all degrees: propositions, attributes of individuals, dyadic relations of individuals, triadic relations of individuals, and so on. Within the calculus distinctions of degree will arise only from formal properties of the elements, just as e.g. the distinction between odd and even arises in arithmetic. The two primitive functions or operations of the calculus, and consequently all derivative functions, admit as arguments all elements without restriction. On adaptation to the calculus any function of traditional logistic will accordingly be so generalized as to admit as arguments any elements whatever, whether of the traditionally appropriate degrees or otherwise, and will hence come to embrace as special cases not only its original but an infinity of other more or less analogous functions of traditional logistic as well.[3]

The exclusion of higher types and bound variables imposes less restriction upon the compass of the conceptual calculus than might appear from the ubiquity of those devices in traditional systematizations of logistic. Although the confinement of the elements to concepts of first type excludes every function which, like predication (membership) or the unit-class operator, demands difference of type between arguments or between value and argument, it does not necessarily exclude any other function, e.g. identity, whose traditional definiens happens to involve type differences: for such a function could always either be given an equivalent definiens lacking that feature, or, at worst, be taken as primitive. Similarly the avoidance of bound variables does not necessarily exclude an expression whose traditional definiens contains bound variables.

This paper will be limited to the definitional side: presentation of the primitive notions and various derivative ones, correlation of them with notions of traditional logistic, and consideration of the range of derivable notions. Attendant argument will proceed informally through application of intuitive logic to the given interpretations of the signs. Formal postulates and rules of inference are deferred.

2. **Preliminary notation.** In metamathematical discussion German letters will be used for unspecified expressions, as with Hilbert. A symbolic passage containing one or more such letters is to be understood as referring to an *expression* of

[3] While this gain in generality includes that described in *Logistic* [Quine, *A system of logistic*], pp. 3, 191–193, the synthesis here undertaken is more basic. Whereas in *Logistic* the cleavage between degrees reduces to difference of type, under the present scheme even this residuum of extra-formal cleavage disappears. Furthermore, though in *Logistic* most functions are released from confinement to arguments of single degrees, they are not extended to arguments of all degrees, like and unlike, positive and zero, as under the present scheme.

the depicted form, rather than to the meaning of such an expression. In the absence of German letters, single quotes will be used as usual to distinguish an expression from its denotation.

In exposition a preliminary logistical language will be needed. At the expository level we shall have to consider not only concepts of first type, but also entities of zero type, viz. sequences of individuals. The former will be represented by conceptual variables 'α', 'β', etc. as in the conceptual calculus itself, and the latter by sequential variables 'X', 'Y', etc. The number of places in a sequence will be called its *length*, and a sequence of length n ($\geqq 0$) will be called an *n-ad*. Monads, or sequences of one place, are simply individuals. There is only one *medad*, or sequence of no places; this follows vacuously from the principle that sequences are identical if homologous places are identical.

In the preliminary language juxtaposition will represent that operation, called *concatenation*, whereby an m-ad X and an n-ad Y are laid end to end in that order so as to form an $(m+n)$-ad XY. In view of the obvious associativity of concatenation,[4] parentheses are suppressed in this connection; elsewhere in the preliminary language they are used in the usual fashion.

For *predication* the notation $\mathfrak{x}\epsilon\mathfrak{y}$ will be used. $X\epsilon\alpha$ is the proposition to the effect that the n-ad X *exhibits* the n-adic concept α; more specifically $X\epsilon\alpha$ is the proposition α itself, or the proposition to the effect that the individual X has the attribute α, or the proposition to the effect that n-ad X stands in the n-adic relation α, according as n is 0, 1, or more. The identity sign '$=$' will be used both between sequential and between conceptual expressions. Such expressions of predication and identity will be built up into further propositional expressions by use of the signs '$\sim$', '$|$', '$\vee$', '$.$', '$\supset$', and '$\equiv$' of the familiar propositional truth-functions, and by use of the following notation of quantification with respect to sequential variables: $(X)\mathfrak{y}$ and $(\exists X)\mathfrak{y}$ will mean respectively that all and that some sequences X of proper length (v. infra) satisfy the condition formulated as the propositional expression $\mathfrak{y}$.

In view of the adoption, throughout the present study, of the extensionality principle

$$(1) \qquad (\alpha = \beta) \equiv (X)((X\epsilon\alpha) \equiv (X\epsilon\beta)),$$

an n-adic concept α is determinate once any condition upon n-ads has been specified which is necessary and sufficient to their exhibition of α. A notation $XY \cdots W \ni \mathfrak{y}$[5] of *abstraction* relative to sequential variables is accordingly adopted to represent the concept exhibited by just those sequences $XY \cdots W$ whose concatenands $X, Y, \cdots$, and W are sequences of proper lengths satisfying the condition $\mathfrak{y}$.

What is meant above by 'proper length' is determined, if at all, relatively to conceptual expressions occurring in $\mathfrak{y}$, on the basis of these *conventions of length and degree:* If a predication $(\mathfrak{x}\epsilon\mathfrak{z})$ occurs in $\mathfrak{y}$, then $\mathfrak{x}$ and $\mathfrak{z}$ are to be interpreted

[4] This operation so contrasts with the related operation of *ordination* used in ***Logistic*** (q.v., pp. 10–16). The present procedure involves no consideration of sequences *of* sequences, with the attendant internal groupings.

[5] This use of '$\ni$', readable as 'such that,' is due to Peano.

in the context $\mathfrak{y}$ as representing sequence and concept of like length and degree; if a sequential identity ($\mathfrak{x}=\mathfrak{z}$) occurs in $\mathfrak{y}$, then $\mathfrak{x}$ and $\mathfrak{z}$ are to be interpreted in $\mathfrak{y}$ as representing sequences of like lengths; and a result of concatenation is equal in length to the sum of the lengths of the concatenands. To whatever extent $\mathfrak{y}$ fails to determine the length of X through these channels, the expressions $(X)\mathfrak{y}$, $(\exists X)\mathfrak{y}$, $X\ni\mathfrak{y}$, $XY\ni\mathfrak{y}$, $YX\ni\mathfrak{y}$, etc. are ambiguous. In the special case where $\mathfrak{y}$ determines X as of length 0, i.e. as the medad, the prefixes '(X)', '$(\exists X)$', and '$X\ni$' are vacuous and suppressible, and the prefixes '$XY\ni$' etc. reduce to '$Y\ni$' etc. In this case the occurrences of 'X' throughout $\mathfrak{y}$ are likewise vacuous and suppressible, as is clear from three considerations: XY and YX are obviously Y (i.e. the medad is the identity element for concatenation); $X\epsilon\alpha$ has been seen to be the proposition α; and $X=\mathfrak{z}$ and $\mathfrak{z}=X$, by compelling $\mathfrak{z}$ likewise to represent the medad in view of the conventions of length and degree, become tautologies.

3. **Primitives.** If we ignore formal definitions, thinking of all abbreviations thus introduced as expanded rather into primitive terms, we can describe the class CC of all significant expressions of the conceptual calculus in the following recursive fashion: Conceptual variables are CC, and if $\mathfrak{x}$ and $\mathfrak{y}$ are CC so are $(\mathfrak{x} \uparrow \mathfrak{y})$ and $(\mathfrak{x} * \mathfrak{y})$. Of the two primitive binary operations of the calculus, expressed thus by arrow and star, the first is the obvious generalization of the similarly denoted operation of *PM*;[6] it is describable thus in terms of the preliminary language:[7]

$$(2) \qquad (\alpha \uparrow \beta) = XY\ni((X\epsilon\alpha).(Y\epsilon\beta)).$$

The degree of the concept $(\alpha \uparrow \beta)$ is thus the sum of those of α and β, and the sequences exhibiting $(\alpha \uparrow \beta)$ are just those which are obtainable by concatenating α-sequences (sequences exhibiting α) with β-sequences in that order. Where in particular α and β are medadic, so that 'X' and 'Y' become vacuous in (2), $(\alpha \uparrow \beta)$ becomes the conjunction of the propositions α and β.

The result $(\alpha * \beta)$ of the other primitive operation is a concept of degree $\max(m-n, 2n-2m)$ where m and n are the respective degrees of α and β. Where $m \geqq n$,

$$(3) \qquad (\alpha * \beta) = X\ni \sim (\exists Y)((XY\epsilon\alpha).(Y\epsilon\beta));$$

i.e., $(\alpha * \beta)$ is exhibited by just those $(m-n)$-ads which are *not* residues of lopping α-sequences off the ends of β-sequences. Where $m=n$, so that 'X' in (3) comes to represent the medad and hence drops out,

$$(4) \qquad (\alpha * \beta) = \sim (\exists Y)((Y\epsilon\alpha).(Y\epsilon\beta));$$

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[6] Whitehead and Russell's ***Principia mathematica***, 2nd. ed., vol. 1, will be thus referred to. For all references to symbols thereof use the index to definitions, pp. 667–668.

[7] This is not the same generalization as occurs in ***Logistic*** (pp. 161–162), for the latter depends upon groupings within sequences. (V. supra, note 4.) Analogous divergence of generalizations results throughout relational theory. The present generalizations differ also quantitatively from those of ***Logistic*** (v. supra, note 3); thus the function described in (2) admits medadic arguments, while the analogue in ***Logistic*** does not. With other functions the difference of scope is greater.

for idemgradual[8] arguments the operation thus yields a proposition, and coincides with the copula of mutual exclusiveness. Where more particularly $m=n=0$, so that 'Y' drops out of (4), $(\alpha * \beta)$ becomes the disjunction[9] of the propositions α and β.

For the cases where $m<n$, not covered by the above, the following interpretation is adopted:

$$(5) \qquad (\alpha * \beta) = XX \ni \sim (\exists Y)((Y \epsilon \alpha) . (XY \epsilon \beta)),$$

which is to say

$$(6) \qquad (\alpha * \beta) = XX \ni (X \epsilon (\beta * \alpha))$$

wherein $(\beta * \alpha)$ is as of (3). This explanation agrees with (3) when extended to the boundary case where $m=n$: for in this case 'X' in (5) comes to represent the medad and hence drops out, reducing (5) to (4). Wherever $m \leqq n$, thus, $(\alpha * \beta)$ is the concept exhibited by just those once-repetitive $(2n-2m)$-ads XX whose halves X exhibit $(\beta * \alpha)$.

4. **Propositions.** Substitution of 'α' for 'β' in (4) shows $(\alpha * \alpha)$ to be $\sim(\exists Y)(Y \epsilon \alpha)$. This provides us with a formal definition of nullity:

D1. $\qquad 0\mathfrak{x} \quad \text{for} \quad (\mathfrak{x} * \mathfrak{x}).$

(In such definitions a German letter represents any CC, and the word 'for' is short for the words 'is adopted as shorthand for.') Now it was observed that where α and β are propositions, $(\alpha * \beta)$ is their disjunction; hence where β is a proposition, 0β or $(\beta * \beta)$ is the self-disjunction or denial of β. In particular, therefore, since 0α is a proposition, 00α is the denial of 0α: the denial of the nullity of α. This leads to the following definition:

D2. $\qquad \exists\mathfrak{x} \quad \text{for} \quad 00\mathfrak{x}.$

'0α' and '$\exists\alpha$' thus affirm respectively that no sequences and that some sequences exhibit α.

In view of (1) and the uniqueness of the medad, there are just two "propositions" or medadic concepts: one exhibited, the other not exhibited, by the medad. By "propositions" here are meant not the formulae but the elements of the calculus of propositions; not sentences, therefore, but hypostatized entities whereof sentences are to be regarded as names.[10] The point of view here adopted is that propositions in this sense are to be distinguished only in point of truth and falsehood; that considerations of import, modality, etc. belong rather at the metamathematical level where the sentences themselves are taken as subject-matter and examined for their verbal constitution or their transformability into other sentences in the light of given rules of inference.[11]

[8] Alike in degree; Sheffer's term.

[9] The term *disjunction* is used here for Sheffer's stroke-function, the denial of conjunction; never for alternation.

[10] See Quine, *Ontological remarks on the propositional calculus*, **Mind**, n.s. vol. 43 (1934), pp. 472–476.

[11] See Frege, ***Grundgesetze der Arithmetik***, vol. 1, p. 50; Carnap, ***Logische Syntax der Sprache***, pp. 38, 192–202.

T and F, describable respectively as the true and the false proposition, or, if the reader prefers, as the truth-values yes and no, enter the calculus through the following definitions:

D3, D4. 'T' for '$(0\alpha * \exists\alpha)$', 'F' for '0T'.

D3 does not mean that 'T' is adopted as shorthand for every expression of the form $(0\mathfrak{x} * \exists\mathfrak{x})$, for that would prejudge the constancy of the function thus expressed; rather is 'T' adopted as shorthand for the specific expression indicated, containing the specific letter 'α'. Whatever postulates are ultimately to be adopted will be such, however, as to guarantee the constancy in question and thus enable us in practice to ignore the implicit argument.[12]

$(0\alpha * \exists\alpha)$, being the disjunction of the propositions 0α and $\exists\alpha$, is true if and only if 0α and $\exists\alpha$ are not both true. But the latter are not both true, for they contradict each other. Thus $(0\alpha * \exists\alpha)$ is, regardless of α, the true proposition; hence the adoption of D3. Now it was seen earlier that where α is a proposition, 0α is its denial; hence the adoption of D4.

The formulae '0α' and '$\exists\alpha$', as defined earlier, represent one or other of the medadic elements or propositions T and F: they represent F and T, or T and F, respectively, according as the concept α is or is not exhibited by at least one sequence. The signs '0' and '$\exists$' thus indicate unitary operations which apply to any concept and yield T or F as result. Applied in particular to medadic concepts, whereas the first of these operations reduces to denial, the second becomes vacuous: 0T is F, 0F is T, $\exists$T is T, and $\exists$F is F.

5. **Further constructs on the star-function.** Of the respective notions $-\alpha$, V, Λ, I, $R``\beta$, D'R, and $\alpha \cap \beta$ of *PM*, obvious generalizations are describable as follows; formal definitions will be withheld for the moment. $\mathfrak{x}_\gamma$ is to be understood as any tautology, e.g. $(\mathfrak{x}\epsilon\gamma) \supset (\mathfrak{x}\epsilon\gamma)$, serving to restrict $\mathfrak{x}$ to the representation of sequences equal in length to the degree of γ.

(7) $$-\alpha = X \ni \sim (X\epsilon\alpha).$$

(8) $$\{\gamma\} = X \ni X_\gamma.$$

(9) $$-\{\gamma\} = X \ni \sim X_\gamma.$$

(10) $$\mathrm{I}\{\gamma\} = XX \ni X_\gamma.$$

(11) $$(\alpha``\beta) = X \ni (\exists Y)((XY\epsilon\alpha) . (Y\epsilon\beta)).$$

(12) $$(\alpha``\{\gamma\}) = X \ni (\exists Y)((XY\epsilon\alpha) . Y_\gamma).$$

(13) $$(\alpha \cap \beta) = X \ni ((X\epsilon\alpha) . (X\epsilon\beta)).$$

(7) and (13) generalize the Boolean functions of negation and composition.[13] Where α and β are m-adic, $-\alpha$ and $(\alpha \cap \beta)$ are the m-adic concepts exhibited respectively by just those m-ads which do not exhibit α, and just those which exhibit both α and β. Now from negation and composition all other Boolean func-

[12] Still, those inclined may think of 'T' as 'T_α' and incorporate the vacuous argument similarly into derivative constants such as 'F'.

[13] C. S. Peirce's terms 'composition' and 'aggregation' are clearly preferable to 'logical multiplication' and 'logical addition.'

tions can be generated in familiar fashion; thus the implex[14] $(\alpha\supset\beta)$ is $-(\alpha \cap -\beta)$, the aggregate[13] $(\alpha \cup \beta)$ is $(-\alpha\supset\beta)$, and the concourse[15] $(\alpha\#\beta)$ is $((\alpha\supset\beta) \cap (\beta\supset\alpha))$. The functions based thus on the generalizations (7) and (13) become, in the special case where α and β are monadic, the operations of the ordinary Boolean calculus of attributes; where α and β are dyadic they become the analogous operations of the dyadic-relational calculus, such as are distinguished from the former in *PM* by a dot applied to the operation-sign; correspondingly for higher degrees. Where α and β are medadic, so that 'X' drops out of (7) and (13), negation and composition reduce to the denial and conjunction of the propositional calculus; aggregation, implexion, and concursion reduce similarly to alternation and material implication and equivalence.

The constant universal and null elements of Boolean algebra become, under the generalizations (8) and (9), unitary functions. $\{\gamma\}$ and $-\{\gamma\}$ are the universal and null concepts idemgradual with γ. Where γ is k-adic they are the k-adic concepts exhibited respectively by all and by no k-ads. Where γ is medadic they become T and F.

(10)–(12) generalize three non-Boolean notions occurring in *PM*'s theory of dyadic relations. Thus generalized, $\mathrm{I}\{\gamma\}$ is the concept exhibited by all once-repetitive $2k$-ads, $(\alpha``\beta)$ is the $(m-n)$-adic concept exhibited by the residues of lopping $\beta$-sequences off the ends of $\alpha$-sequences, and $(\alpha``\{\gamma\})$ is the concept exhibited by those $(m-k)$-ads which are initial to α-sequences, where m, n, and k are the respective degrees of α, β, and γ. The three notions reduce to the I, $R``\beta$, and $\mathrm{D}`R$ of *PM* when $m=2$ and $n=k=1$. In the degenerate case where $n=k=0$, $\mathrm{I}\{\gamma\}$ reduces to T, $(\alpha``\{\gamma\})$ to α, and $(\alpha``\beta)$ to $\alpha$ or $-\{\alpha\}$ according as $\beta$ is T or F. Where $m=n=k$, $(\alpha``\beta)$ and $(\alpha``\{\gamma\})$ become the respective propositions $\exists(\alpha \cap \beta)$ and $\exists\alpha$. Where more particularly $m=n=0$, $(\alpha``\beta)$ becomes, like $(\alpha \cap \beta)$ and $(\alpha \uparrow \beta)$, the conjunction of the propositions α and β.

The notations described in (7)–(13) enter the conceptual calculus through these definitions:

D5. $-\mathfrak{x}$ for $(\mathfrak{x} \ast \mathrm{T})$.

D6. $\{\mathfrak{x}\}$ for $(\mathfrak{x} \ast \mathrm{F})$.

D7. $\mathrm{I}\mathfrak{x}$ for $(\mathrm{T} \ast -\mathfrak{x})$.

D8. $(\mathfrak{x}``\mathfrak{y})$ for $-(\mathfrak{x} \ast \mathfrak{y})$.

D9. $(\mathfrak{x} \cap \mathfrak{y})$ for $(\mathrm{I}\mathfrak{x}``\mathfrak{y})$.

Since the degree 0 of T and F cannot exceed the degree of α, explanation of $(\alpha \ast \mathrm{T})$ and $(\alpha \ast \mathrm{F})$ calls for (3) rather than (5). In both cases 'Y' in (3) comes to represent the medad, and drops out; $-\alpha$ and $\{\alpha\}$ as defined in D5 and D6 are thus $X\ni\sim((X\epsilon\alpha).\mathrm{T})$ and $X\ni\sim((X\epsilon\alpha).\mathrm{F})$ respectively. Since the conjunction '$(X\epsilon\alpha).\mathrm{T}$' is equivalent to '$X\epsilon\alpha$' alone, (7) is verified; and since the conjunction '$(X\epsilon\alpha).\mathrm{F}$' represents F regardless of whether or not $X\epsilon\alpha$, (8) is likewise verified. (9), then, follows directly from (7) and (8).

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[14] See *Logistic*, pp. 91–92.

[15] See *Logistic*, pp. 125, 127.

Since the degree of T cannot exceed that of $-\delta$, $(\mathrm{T} * -\delta)$ is to be explained according to (6), i.e. as $XX\ni(X\epsilon(-\delta * \mathrm{T}))$. But, in view of D5, $(-\delta * \mathrm{T})$ is $--\delta$, or δ; hence

$$\mathrm{I}\delta = XX\ni(X\epsilon\delta) \tag{14}$$

under D7. $\mathrm{I}\delta$ is thus exhibited by just those sequences whose halves are alike and exhibit δ. (10), which is less general than (14), proceeds from (14) in view of (8) when we take δ as a universal concept $\{\gamma\}$.

(11) makes sense only where the degree m of α is not less than the degree n of β; on the other hand D8, like all definitions of the conceptual calculus, knows no restriction of degree. But throughout the scope of (11), i.e. wherever $m \geqq n$, (7) and (3) show that D8 defines $(\alpha``\beta)$ in conformity with (11). The generalization D8 thus includes the generalization (11); the added cases, where $m < n$, may come out as they will. Actually, wherever $m \leqq n$, (7) and (5) show $(\alpha``\beta)$ as of D8 to be exhibited by just those $(2n-2m)$-ads Z which are not of the form XX where $\sim(\exists Y)((Y\epsilon\alpha).(XY\epsilon\beta))$; i.e.,

$$(\alpha``\beta) = Z\ni\sim(\exists X)((Z = XX).\sim(\exists Y)((Y\epsilon\alpha).(XY\epsilon\beta))). \tag{15}$$

(12) makes sense only where the degree of γ does not exceed that of α. That (12) is satisfied by the definitions, throughout its range, follows from the fact that (11) is satisfied throughout the corresponding range: for (12) comes of (11) by taking β as $\{\gamma\}$. What $(\alpha``\{\gamma\})$ may be outside the range of (12) is found by substitution in (15).

Whereas (13) makes sense only where the degree m of α equals the degree n of β, D9 applies regardless of degrees. Where the degree $2m$ of $\mathrm{I}\alpha$ is not less than n, (11) shows $(\alpha \cap \beta)$ as of D9 to be $X\ni(\exists Y)((XY\epsilon\mathrm{I}\alpha).(Y\epsilon\beta))$. But, by (14), $XY\epsilon\mathrm{I}\alpha$ if and only if XY is of the form WW where $W\epsilon\alpha$. Thus

$$(\alpha \cap \beta) = X\ni(\exists Y)(\exists W)((XY = WW).(W\epsilon\alpha).(Y\epsilon\beta)). \tag{16}$$

Where on the other hand $2m \leqq n$, (15) replaces (11) in the above reasoning; thus

$$(\alpha \cap \beta) = Z\ni\sim(\exists X)((Z = XX).\sim(\exists Y)(\exists W)((Y = WW).(W\epsilon\alpha).(XY\epsilon\beta)))$$

$$= Z\ni\sim(\exists X)((Z = XX).\sim(\exists W)((W\epsilon\alpha).(XWW\epsilon\beta))). \tag{17}$$

In the special case where $m = n$, so that '$(W\epsilon\alpha).(Y\epsilon\beta)$' determines '$W$' and '$Y$' as representing sequences of the same length, '$XY = WW$' reduces to '$(X = W).(Y = W)$'; therewith (16) reduces as follows:

$$(\alpha \cap \beta) = X\ni(\exists Y)(\exists W)((X = W).(Y = W).(W\epsilon\alpha).(Y\epsilon\beta))$$
$$= X\ni((X\epsilon\alpha).(X\epsilon\beta)).$$

D9, though much more general than (13), is thus seen to yield (13) throughout the range where (13) applies.

Whereas in the familiar cases where $m = n$ composition is commutative and yields an m-adic concept, in the general case composition as defined in D9 is not commutative and yields a $\max(2m-n,\ 2n-4m)$-adic concept. Where $2m = n$, (16) and (17) both show $(\alpha \cap \beta)$ to be a proposition, viz. $(\exists W)((W\epsilon\alpha).(WW\epsilon\beta))$.

The generality achieved in D9 carries over to implexion and the other Boolean functions, defined as they are in terms of negation and composition. What those derivative operations yield in the unaccustomed cases of aliogradual[16] arguments is determinable from the above examination of composition.

6. **Inclusion and identity.** Parallel to the generalizations (7) and (13) of negation and composition, obvious generalizations of the Boolean copulas or binary predicates of inclusion and identity are as follows:

(18) $$(\alpha \subset \beta) = (X)((X \epsilon \alpha) \supset (X \epsilon \beta)).$$

(19) $$(\alpha = \beta) = (X)((X \epsilon \alpha) \equiv (X \epsilon \beta)).$$

Similar generalization of the unitary Boolean predicates of nullity and non-nullity was afforded by D1 and D2. Ordinarily in Boolean algebra the predicates are not reckoned among the functions, or operations, for they yield propositions rather than elements of the algebra. Since on the other hand propositions are included among the elements of the conceptual calculus, the predicates here become ordinary operations, as already remarked in connection with D1 and D2. They have the peculiarity, among operations, of yielding always one or other of the special elements T and F.

In the special case where the arguments α and β are medadic, so that 'X' drops out of (18) and (19), inclusion and identity reduce to the material implication and equivalence of the propositional calculus—just as nullity was seen to reduce to denial. Thus the truth-functions of denial and material implication and equivalence are the medadic applications not only of the respective Boolean operations of negation, implexion, and concursion, as seen earlier, but also of the respective Boolean predicates of nullity, inclusion, and identity. The analogous is true of every truth-function; each is the medadic case both of a Boolean operation and of at least one (actually more than one) Boolean predicate. Applied to k-adic concepts, the Boolean operations and predicates are distinguished in that the former yield k-adic concepts, the latter medadic ones; in the propositional calculus this cleavage vanishes, for here $k=0$.

Inclusion and identity are defined thus for the conceptual calculus:

D10. $$(\mathfrak{x} \subset \mathfrak{y}) \quad \text{for} \quad 0 - \mathrm{I}(\mathfrak{x} * - \mathfrak{y}).$$

D11. $$(\mathfrak{x} = \mathfrak{y}) \quad \text{for} \quad ((\mathfrak{x} \subset \mathfrak{y}) \cap (\mathfrak{y} \subset \mathfrak{x})).$$

$(\alpha \subset \beta)$ is thus the proposition $0 - \mathrm{I}(\alpha * -\beta)$ to the effect that no sequence exhibits $-\mathrm{I}(\alpha * -\beta)$, i.e. that every sequence of appropriate length exhibits $\mathrm{I}(\alpha * -\beta)$. Thus

(20) $$(\alpha \subset \beta) = (Z)(Z \epsilon \mathrm{I}(\alpha * - \beta)).$$

Now consider the only cases where (18) makes sense, viz. where the degrees m and n of α and β are equal. Here the $\max(m-n,\ 2n-2m)$-adic concept $(\alpha * -\beta)$ becomes medadic. But where δ is medadic, 'X' in (14) comes to represent the medad and drops out; $\mathrm{I}\delta$ reduces to δ. Thus, where $m=n$, $\mathrm{I}(\alpha * -\beta)$ is the medadic concept $(\alpha * -\beta)$; 'I' drops out of (20). Thereupon 'Z' in (20) comes to

[16] Not idemgradual; Sheffer's term.

represent the medad, and drops out in turn. (20) thus reduces to an identification of $(\alpha \subset \beta)$ with $(\alpha * -\beta)$. In view of (4), then,

$$(\alpha \subset \beta) = \sim (\exists Y)((Y \epsilon \alpha) . (Y \epsilon - \beta)),$$

i.e., by (7),
$$(\alpha \subset \beta) = \sim (\exists Y)((Y \epsilon \alpha) . \sim (Y \epsilon \beta)),$$

which is to say (18). Thus D10, though more general than (18) in that it defines $(\alpha \subset \beta)$ not only for idemgradual but also for aliogradual arguments, agrees with (18) where the arguments are idemgradual.

Since the truth-functions issue as special cases of the generalized Boolean operations, viz. those cases where the arguments are propositions, there is no need to adopt the truth-function signs for the conceptual calculus; their purpose is served by the signs of the general Boolean operations with medadic arguments. (Alternatively the Boolean predicates can be used, for the truth-functions are cases likewise of these.) In D11 composition so figures, viz. as the truth-function of conjunction, for its arguments are propositions. Thus D11 defines identity as mutual inclusion. Now where inclusion and identity are as in (18) and (19), mutual inclusion is obviously identity; thus, since D10 conforms to (18), it follows that D11 conforms to (19).

What D10 and D11 yield in cases outside the scope of (18) and (19), viz. under aliogradual arguments, turns upon the question of the multiplicity of individuals. Individuals are foreign to the formal system, but they are basic to the present expository approach in that the elements of the system, concepts, are being thought of as determined by sequences of individuals which "exhibit" them. As to the total number of individuals, it is desirable to assume that there are at least two; this merely excludes the trivial alternative that there be but two sequences of each length and hence but two concepts, the universal and the null, of each degree. (In formal postulates for the conceptual calculus the assumption would of course appear only indirectly, through its effects upon concepts.) Now for a $2k$-ad to lack the repetitive form XX it is sufficient that the first and $(k+1)$-st places of the $2k$-ad be occupied by distinct individuals; hence, granted our assumption that distinct individuals exist, there are $2k$-ads lacking the form XX wherever $k>0$. Since, where δ is k-adic, all Iδ-sequences are $2k$-ads of the form XX, it follows *a fortiori* that where $k>0$ there will be $2k$-ads not exhibiting Iδ. But the degree $\max(m-n,\ 2n-2m)$ of $(\alpha * -\beta)$ exceeds 0 wherever $m \neq n$. Thus, wherever $m \neq n$, not all sequences of appropriate length will exhibit $\mathrm{I}(\alpha * -\beta)$; $(\alpha \subset \beta)$ as described in (20) will consequently be F. Wherever $(\alpha \subset \beta)$ is F, $(\alpha = \beta)$ will also be F, since $(\alpha = \beta)$ is defined as a conjunction upon $(\alpha \subset \beta)$. Thus $(\alpha \subset \beta)$ and $(\alpha = \beta)$ are both F wherever α and β are aliogradual.

To sum up, $(\alpha \subset \beta)$ and $(\alpha = \beta)$ as defined in D10 and D11 are the propositions described in (18) and (19) wherever α and β are idemgradual, and are the false proposition otherwise. $(\alpha \subset \beta)$ is T or F according as α and β are or are not idemgradual concepts such that $(X)((X \epsilon \alpha) \supset (X \epsilon \beta))$, and $(\alpha = \beta)$ is T or F according as α and β are or are not idemgradual concepts such that $(X)((X \epsilon \alpha) \equiv (X \epsilon \beta))$.

Application of the notions of Boolean algebra to aliogradual arguments might be regarded as so remote from the scope of ordinary usage as to be untouched by considerations of analogy; once the obvious generalizations (13), (18), and

(19) are satisfied, we might be content to let further cases of $(\alpha \cap \beta)$, $(\alpha \subset \beta)$, and $(\alpha = \beta)$ turn out as they will. This attitude was adopted in dealing with composition. If it had been adopted also for inclusion and identity, definition would have been easier; $(\alpha \subset \beta)$ could have been defined simply as $(\alpha * -\beta)$, rather than as in D10, since $(\alpha \subset \beta)$ reduces to $(\alpha * -\beta)$ anyway where $m = n$. But if the definition of inclusion were thus simplified, $(\alpha = \beta)$ as defined in D11 would cease to be a proposition and become a certain non-medadic concept when $m \neq n$. Though that result is strictly defensible relatively to existing usage, which neither affirms nor denies identity among aliogradual concepts, still the intuitive sense of identity is better suited if $(\alpha = \beta)$ is a proposition regardless of degrees: if $(\alpha = \beta)$ is true where α and β are the same concept, false wherever they are not, and hence false where they are aliogradual. This is fulfilled by the present procedure.

7. **Degree.** Since there is just one universal concept of each degree, $(\{\alpha\} = \{\beta\})$ will be T if α and β are idemgradual; and otherwise it will be F, since $(\gamma = \delta)$ is F wherever γ and δ are aliogradual. Hence '$(\{\alpha\} = \{\beta\})$' expresses the proposition that α and β are idemgradual. Likewise '$(\{\alpha\} = \mathrm{T})$' expresses the proposition that α is medadic. Indeed, it is convenient to identify universal concepts with degrees—to read '$\{\alpha\}$' as 'the degree of α'; '$(\{\alpha\} = \{\beta\})$' and '$(\{\alpha\} = \mathrm{T})$' then become direct identifications of the degree $\{\alpha\}$ with a degree $\{\beta\}$ and with the zero degree T. Within the conceptual calculus a rudimentary arithmetic of degrees thus arises, in the form of a calculus of universal concepts.

The proposition that the degree m of α is less than the degree n of β can be given formal expression as '$\exists(\alpha'' - \{\beta\})$', as will now be shown. If $m \geqq n$, (11) tells us that

$$(\alpha'' - \{\beta\}) = X \ni (\exists Y)((XY \epsilon \alpha) . (Y \epsilon - \{\beta\}))$$

and hence that for a sequence X to exhibit $(\alpha'' - \{\beta\})$ there must be a sequence Y fulfilling the impossible condition '$Y \epsilon - \{\beta\}$'. The proposition $\exists(\alpha'' - \{\beta\})$ is thus F where $m \geqq n$. Now suppose that $m < n$. By (15),

$$(21) \qquad (\alpha'' - \{\beta\}) = Z \ni \sim (\exists X)((Z = XX) . \sim (\exists Y)((Y \epsilon \alpha) . (XY \epsilon - \{\beta\}))).$$

The context '$(Y \epsilon \alpha) . (XY \epsilon - \{\beta\})$', in requiring '$Y$' and '$XY$' to represent m-ads and n-ads respectively, requires 'X' to represent $(n-m)$-ads. $(\alpha'' - \{\beta\})$ as of (21) is then exhibited by just those $(2n-2m)$-ads Z for which there is no $(n-m)$-ad X fulfilling '$Z = XX$' and a certain further condition; *a fortiori*, therefore, $(\alpha'' - \{\beta\})$ is exhibited by all $(2n-2m)$-ads which lack the repetitive form XX altogether. As seen in §6, there will be such $(2n-2m)$-ads provided that $2n-2m > 0$; and this provision is afforded by our hypothesis that $m < n$. We see therefore that there are sequences exhibiting $(\alpha'' - \{\beta\})$, and hence that $\exists(\alpha'' - \{\beta\})$ is T, wherever $m < n$. $\exists(\alpha'' - \{\beta\})$ is thus T or F according as the degree of α is or is not less than that of β.

Treating universal concepts as degrees, then, we can read '$\exists(\alpha'' - \{\beta\})$' as affirming that $\{\alpha\} < \{\beta\}$. By way of elaborating the arithmetic of degrees begun above, we might rewrite '$\exists(\alpha'' - \{\beta\})$' accordingly. This is accomplished by the following definition:

D12. $(\mathfrak{x} < \mathfrak{y})$ for $\exists(\mathfrak{x}``-\mathfrak{y})$.

This explains $(\{\alpha\} < \{\beta\})$ as $\exists(\{\alpha\}``-\{\beta\})$ rather than $\exists(\alpha``-\{\beta\})$; but the two are equivalent, in view of the idemgraduity of $\{\alpha\}$ with α. The nature of the by-products of D12, propositions $(\gamma < \delta)$ where γ and δ are not "degrees" or universal concepts, need not concern us.

$(\{\alpha\} \uparrow \{\beta\})$ is readily found to be the universal concept of degree equal to the sum of those of α and β; hence when we take degrees as universal concepts the arrow figures as a sign of addition. The copulas '$\geqq$', '$>$', and '$\leqq$' can be introduced on the basis of D12 by denial and interchange of arguments. We thus have an arithmetic of degrees containing zero, addition, and the various copulas of equality and inequality. The function $\max(m-n, 2n-2m)$ is also at hand, for $(\{\alpha\}``\{\beta\})$ is readily found to be the degree (universal concept) so related to the degrees $\{\alpha\}$ and $\{\beta\}$. But absolute numerical specification of degree, zero excepted, is presumably impossible within the conceptual calculus.

8. **Functions involving both primitives.** Definitions thus far have depended only upon the star-function. We proceed now to six representative notions whose definitions presuppose both primitives.

Of the $R \upharpoonright \beta$, $\breve{R}$, $Q|R$, and $\breve{R}``\beta$ of *PM*, obvious generalizations are as follows:

(22) $$(\alpha \upharpoonright \beta) = XY\ni((XY\epsilon\alpha).(Y\epsilon\beta)).$$

(23) $$\mathrm{Cnv}\{\gamma\}\alpha = XY\ni((YX\epsilon\alpha).Y_\gamma).$$

(24) $$(\alpha\{\gamma\}\beta) = XY\ni(\exists Z)((XZ\epsilon\alpha).(ZY\epsilon\beta).Z_\gamma).$$

(25) $$(\alpha_{,,}\beta) = Y\ni(\exists X)((XY\epsilon\alpha).(X\epsilon\beta)).$$

The analogous generalization of the Q'R of *PM* proceeds from (25) when β is taken as $\{\gamma\}$, just as the generalization (12) of D'R proceeds from (11).

For the ungeneralized cases as of *PM* γ is monadic throughout the above, α is dyadic, and β is dyadic in (24) and monadic elsewhere. The generalizations are readily seen to have the following degenerate cases: $\mathrm{Cnv}\{\gamma\}\alpha$ reduces to α when γ is medadic or idemgradual with α; $(\alpha \upharpoonright \beta)$ and $(\alpha_{,,}\beta)$ behave like $(\alpha``\beta)$ (see §5) when $\beta$ is medadic, and reduce respectively to $(\alpha \cap \beta)$ and $\exists(\alpha \cap \beta)$ when $\beta$ is idemgradual with $\alpha$; and $(\alpha\{\gamma\}\beta)$ reduces to $(\alpha \uparrow \beta)$, $(\beta_{,,}\alpha)$, or $(\alpha``\beta)$ according as γ is medadic, idemgradual with α, or idemgradual with β.

The notions generalized in (10)–(12) and (22)–(25) are confined in *PM* to the theory of dyadic relations. Further notions which, had they traditionally been studied at all, would have been confined to the theories of relations of the successive higher degrees, admit of similar generalization. Two such are the unitary operations which, ungeneralized, apply to a triadic relation β and yield respectively the dyadic relation Wβ exhibited by just those dyads (x, y) such that $(x, y, y)\epsilon\beta$, and the triadic relation Sβ exhibited by just those triads (x, y, z) such that $(x, z, y)\epsilon\beta$. The generalizations are as follows:

(26) $$\mathrm{W}\{\gamma\}\beta = XY\ni((XYY\epsilon\beta).Y_\gamma).$$

(27) $$\mathrm{S}\{\alpha\}\{\gamma\}\beta = XYZ\ni((XZY\epsilon\beta).(ZY)_\alpha.Y_\gamma).$$

The notions described in (22)–(27) enter the conceptual calculus through the following definitions:

D13.	$(\mathfrak{x} \upharpoonright \mathfrak{y})$	for	$(\mathfrak{x} \cap ((\mathfrak{x}``\mathfrak{y}) \uparrow \mathfrak{y}))$.
D14.	$\mathrm{Cnv}\mathfrak{z}\mathfrak{x}$	for	$((\mathrm{I}\{(\mathfrak{x} \uparrow \mathfrak{z})\}``\mathrm{I}\mathfrak{z})``\mathfrak{x})$.
D15.	$(\mathfrak{x}\mathfrak{z}\mathfrak{y})$	for	$\mathrm{Cnv}\{(\mathfrak{y}``\mathfrak{z})\}(\mathrm{Cnv}(\mathfrak{x} \uparrow \mathfrak{z})(\mathfrak{x} \uparrow \mathfrak{y})``\mathrm{I}\mathfrak{z})$.
D16.	$(\mathfrak{x},,\mathfrak{y})$	for	$(\mathfrak{y}\mathfrak{y}\mathfrak{x})$.
D17.	$\mathrm{W}\mathfrak{z}\mathfrak{y}$	for	$((\mathfrak{y} \upharpoonright \mathrm{I}\mathfrak{z})``\mathfrak{z})$.
D18.	$\mathrm{S}\mathfrak{x}\mathfrak{z}\mathfrak{y}$	for	$(\mathfrak{y}\mathfrak{x}(\mathrm{I}\{(\mathfrak{x} \uparrow \mathfrak{z})\}``\mathrm{I}\mathfrak{z}))$.

D13–D18 are more general than (22)–(27), for whereas D13–D18 know no restrictions of degree, on the other hand (22) and (25) make sense only where $m \geqq n$, (23) only where $m \geqq k$, (24) only where $m \geqq k \leqq n$, (26) only where $n \geqq 2k$, and (27) only where $n \geqq m \geqq k$, if m, n, and k are the respective degrees of α, β, and γ. D14, D15, D17, and D18 outrun (23), (24), (26), and (27) also in the further respect of yielding general functions $\mathrm{Cnv}\delta\alpha$, $(\alpha\delta\beta)$, $\mathrm{W}\delta\beta$, and $\mathrm{S}\zeta\delta\beta$, whereof the $\mathrm{Cnv}\{\gamma\}\alpha$, $(\alpha\{\gamma\}\beta)$, $\mathrm{W}\{\gamma\}\beta$, and $\mathrm{S}\{\alpha\}\{\gamma\}\beta$ of (23), (24), (26), and (27) represent only those special cases where δ and ζ are universal concepts $\{\gamma\}$ and $\{\alpha\}$. It will now be shown that D13–D18 conform to (22)–(27) throughout the ranges of applicability of the latter. What D13–D18 yield for degrees outside the scope of (22)–(27) will not be entered upon,[17] but is determinable in routine fashion.

Let $m \geqq n$, where α is m-adic and β n-adic. (2) and (11) show that

$$(28) \qquad ((\alpha``\beta) \uparrow \beta) = XY \ni ((\exists Z)((XZ \epsilon \alpha) . (Z \epsilon \beta)) . (Y \epsilon \beta)).$$

Since the degree of $(\alpha``\beta)$ is $m-n$, that of $((\alpha``\beta) \uparrow \beta)$ is $m-n+n$, or m. Hence the compound $(\alpha \cap ((\alpha``\beta) \uparrow \beta))$ falls under (13), rather than requiring (16) or (17) for its interpretation; i.e., it is exhibited by just those $m$-ads which exhibit $\alpha$ and $((\alpha``\beta) \uparrow \beta)$. Thus, in view of (28),

$$(29) \qquad (\alpha \cap ((\alpha``\beta) \uparrow \beta)) = XY \ni ((XY \epsilon \alpha) . ((\exists Z)((XZ \epsilon \alpha) . (Z \epsilon \beta)) . (Y \epsilon \beta))).$$

The clause '$(\exists Z)((XZ\epsilon\alpha) . (Z\epsilon\beta))$' follows from '$XY\epsilon\alpha$' and '$Y\epsilon\beta$', and is therefore redundant. Its deletion reduces the right side of (29) to that of (22), and thus shows D13 to yield (22) wherever $m \geqq n$.

Let α be m-adic and δ k-adic, so that $(\alpha \uparrow \delta)$ is $(m+k)$-adic. By (10), then, $\mathrm{I}\{(\alpha \uparrow \delta)\}$ is exhibited by just those $2(m+k)$-ads having the form XX, or, expressed in terms of individuals, the form

$$(30) \qquad (x_1, \cdots x_{m+k}, x_1, \cdots x_{m+k}).$$

Now since $\mathrm{I}\delta$ is exhibited by just those $2k$-ads YY such that $Y\epsilon\delta$, (30) will end in an $\mathrm{I}\delta$-sequence if and only if the last k-ad $(x_{m+1}, \cdots x_{m+k})$ of (30) exhibits δ and duplicates the preceding k-ad—which is $(x_{m-k+1}, \cdots x_m)$, if $m \geqq k$. Thus, if $m \geqq k$, (30) ends in an $\mathrm{I}\delta$-sequence if and only if (30) is

[17] In this direction one interesting point might be mentioned, viz. that $\mathrm{Cnv}\delta\alpha$ and α are idemgradual regardless of the degree of δ.

$$(x_1, \cdots x_m, x_{m-k+1}, \cdots x_m, x_1, \cdots x_m, x_{m-k+1}, \cdots x_m)$$

and $(x_{m-k+1}, \cdots x_m)\epsilon\delta$. When the I$\delta$-sequence is lopped off the end, the residue is

(31) $$(x_1, \cdots x_m, x_{m-k+1}, \cdots x_m, x_1, \cdots x_{m-k}).$$

Thus $(\mathrm{I}\{\alpha \upharpoonright \delta)\}``\mathrm{I}\delta)$, exhibited as it is by just those sequences which are residues of lopping Iδ-sequences off the ends of $\mathrm{I}\{(\alpha \upharpoonright \delta)\}$-sequences (*cf.* §5), will be exhibited by just those $2m$-ads (31) such that $(x_{m-k+1}, \cdots x_m)\epsilon\delta$. Now the residues of lopping α-sequences off the ends of these $2m$-ads (31), in turn, will be just those m-ads $(x_1, \cdots x_m)$ such that $(x_{m-k+1}, \cdots x_m)\epsilon\delta$, as before, and such further that the m-ad $(x_{m-k+1}, \cdots x_m, x_1, \cdots x_{m-k})$ exhibits α. Cnv$\delta\alpha$, defined as it is in D14 as $((\mathrm{I}\{(\alpha \upharpoonright \delta)\}``\mathrm{I}\delta)``\alpha)$, is exhibited by just such residues $(x_1, \cdots x_m)$; hence, using 'X' for '$(x_1, \cdots x_{m-k})$' and 'Y' for '$(x_{m-k+1}, \cdots x_m)$',

(32) $$\mathrm{Cnv}\delta\alpha = XY\ni((Y\epsilon\delta).(YX\epsilon\alpha)).$$

This explains Cnv$\delta\alpha$ wherever, as assumed above, the degrees m and k of α and δ are such that $m \geqq k$. Taking δ in (32) as $\{\gamma\}$, we have (23).

Let α be m-adic, β n-adic, and δ k-adic. Then $(\alpha \upharpoonright \beta)$ is exhibited by just those $(m+n)$-ads $(x_1, \cdots x_{m+n})$ such that $(x_1, \cdots x_m)\epsilon\alpha$ and $(x_{m+1}, \cdots x_{m+n})\epsilon\beta$, and $(\alpha \upharpoonright \delta)$ is exhibited by just those $(m+k)$-ads $(x_1, \cdots x_{m+k})$ such that $(x_1, \cdots x_m)\epsilon\alpha$ and $(x_{m+1}, \cdots x_{m+k})\epsilon\delta$. Hence, where $n \geqq k$, an $(\alpha \upharpoonright \beta)$-sequence $(x_1, \cdots x_{m+n})$ will begin in an $(\alpha \upharpoonright \delta)$-sequence if and only if, over and above the fact that $(x_1, \cdots x_m)\epsilon\alpha$ and $(x_{m+1}, \cdots x_{m+n})\epsilon\beta$, $(x_{m+1}, \cdots x_{m+k})\epsilon\delta$. If the $(\alpha \upharpoonright \delta)$-sequence be transferred from initial to terminal position in the $(\alpha \upharpoonright \beta)$-sequence, the resulting sequence is

(33) $$(x_{m+k+1}, \cdots x_{m+n}, x_1, \cdots x_{m+k}).$$

Now if λ be Cnv$(\alpha \upharpoonright \delta)(\alpha \upharpoonright \beta)$, (32) shows λ to be exhibited by just such results of permutation, hence just those $(m+n)$-ads (33) such that $(x_1, \cdots x_m)\epsilon\alpha$, $(x_{m+1}, \cdots x_{m+n})\epsilon\beta$, and $(x_{m+1}, \cdots x_{m+k})\epsilon\delta$. Since each such $(m+n)$-ad ends thus in a δ-sequence $(x_{m+1}, \cdots x_{m+k})$, any such $(m+n)$-ad will end in an Iδ-sequence granted one further condition, viz. that the terminal k-ad $(x_{m+1}, \cdots x_{m+k})$ duplicate the preceding k-ad—which, if $m \geqq k$, is $(x_{m-k+1}, \cdots x_m)$. Thus, where $m \geqq k \leqq n$, the λ-sequences ending in Iδ-sequences comprise just those $(m+n)$-ads (33) such that $(x_1, \cdots x_m)\epsilon\alpha$, $(x_{m+1}, \cdots x_{m+n})\epsilon\beta$, and $(x_{m+1}, \cdots x_{m+k})\epsilon\delta$, as before, and such further that $(x_{m+1}, \cdots x_{m+k})$ is $(x_{m-k+1}, \cdots x_m)$; in other words, just those $(m+n)$-ads

$$(x_{m+k+1}, \cdots x_{m+n}, x_1, \cdots x_m, x_{m-k+1}, \cdots x_m)$$

such that

(34) $$((x_1, \cdots x_m)\epsilon\alpha).((x_{m-k+1}, \cdots x_m, x_{m+k+1}, \cdots x_{m+n})\epsilon\beta).((x_{m-k+1}, \cdots x_m)\epsilon\delta).$$

The residues of lopping the Iδ-sequences off the ends of these $(m+n)$-ads will then comprise just those $(m+n-2k)$-ads

(35) $$(x_{m+k+1}, \cdots x_{m+n}, x_1, \cdots x_{m-k})$$

which, for some $(x_{m-k+1}, \cdots x_m)$, satisfy (34). Just such $(m+n-2k)$-ads, then, are the $(\lambda``\mathrm{I}\delta)$-sequences. Now $(\alpha\delta\beta)$ is defined in D15 as $\mathrm{Cnv}\{(\beta``\delta)\}(\lambda``\mathrm{I}\delta)$, which, since $(\beta``\delta)$ is $(n-k)$-adic, is shown by (23) to be the concept exhibited by just those sequences which arise from $(\lambda``\mathrm{I}\delta)$-sequences by a shift of an $(n-k)$-ad from initial to terminal position therein. But such a shift turns (35) into

$$(36) \qquad (x_1, \cdots x_{m-k}, x_{m+k+1}, \cdots x_{m+n}).$$

Thus $(\alpha\delta\beta)$ is the concept exhibited by just those sequences (36) such that, for some $(x_{m-k+1}, \cdots, x_m)$, (34) holds; i.e., using 'X' for '$(x_1, \cdots x_{m-k})$', 'Y' for '$(x_{m+k+1}, \cdots x_{m+n})$', and '$Z$' for '$(x_{m-k+1}, \cdots x_m)$',

$$(37) \qquad (\alpha\delta\beta) = XY\ni(\exists Z)((XZ\epsilon\alpha).(ZY\epsilon\beta).(Z\epsilon\delta)).$$

This explains $(\alpha\delta\beta)$ wherever, as assumed above, $m \geqq k \leqq n$. Taking δ as $\{\gamma\}$, we have (24).

Where $m \geqq n$, (37) tells us that

$$(\beta\beta\alpha) = XY\ni(\exists Z)((XZ\epsilon\beta).(ZY\epsilon\alpha).(Z\epsilon\beta)).$$

But 'X' drops out as vacuous, compelled as it is by '$XZ\epsilon\beta$' and '$Z\epsilon\beta$' to represent the medad. Omitting also the repetition of '$Z\epsilon\beta$' induced by the deletion of 'X', we find that

$$(\beta\beta\alpha) = Y\ni(\exists Z)((ZY\epsilon\alpha).(Z\epsilon\beta)).$$

Since $(\beta\beta\alpha)$ is $(\alpha,,\beta)$ according to D16, this gives us (25).

Where β is n-adic and δ k-adic, let $n \geqq 2k$. In view of (22), $(\beta \upharpoonright \mathrm{I}\delta)$ is exhibited by just those n-ads which exhibit β and end in $\mathrm{I}\delta$-sequences, i.e., in $2k$-ads YY such that $Y\epsilon\delta$. Hence the $(\beta \upharpoonright \mathrm{I}\delta)$-sequences are just those n-ads XYY such that $XYY\epsilon\beta$ and $Y\epsilon\delta$. Each such sequence thus ends in a δ-sequence Y, which, when lopped off, leaves XY. But $\mathrm{W}\delta\beta$, as defined in D17, is exhibited by just such residues of lopping δ-sequences off the ends of $(\beta \upharpoonright \mathrm{I}\delta)$-sequences. Thus

$$(38) \qquad \mathrm{W}\delta\beta = XY\ni((XYY\epsilon\beta).(Y\epsilon\delta))$$

wherever $n \geqq 2k$. Taking δ in particular as $\{\gamma\}$, we have (26).

Where ζ is m-adic, β n-adic, and δ k-adic, let $n \geqq m \geqq k$. As seen earlier, $(\mathrm{I}\{(\zeta \uparrow \delta)\}``\mathrm{I}\delta)$ is exhibited by just those $2m$-ads (31) such that $(x_{m-k+1}, \cdots x_m)\epsilon\delta$, i.e. just those $2m$-ads $ZYYZ$ such that $Y\epsilon\delta$. Now D18 identifies $\mathrm{S}\zeta\delta\beta$ with the concept $(\beta\zeta(\mathrm{I}\{(\zeta \uparrow \delta)\}``\mathrm{I}\delta))$, which, by (37), is exhibited by just those sequences XW such that, for some m-ad V exhibiting ζ, $XV\epsilon\beta$ and VW is one of the described $2m$-ads. But where the $2m$-ad VW is $ZYYZ$, the m-ad V is ZY; W is then YZ, XW is XYZ, and XV is XZY. Thus

$$(39) \qquad \mathrm{S}\zeta\delta\beta = XYZ\ni((XZY\epsilon\beta).(ZY\epsilon\zeta).(Y\epsilon\delta))$$

wherever $n \geqq m \geqq k$. Taking ζ as $\{\alpha\}$ and δ as $\{\gamma\}$, we have (27).

9. **Anzahl.** It is possible within the conceptual calculus to define, for any given natural number n, the proposition $n\alpha$ to the effect that there are just n α-sequences. Where $n=0$ the definition is D1. Where $n>0$ the definition depends upon $\mathrm{J}_n\alpha$, which, for m-adic α, is the nm-adic concept exhibited by just those

nm-ads $X_1X_2 \cdots X_n$ such that the m-ads X_i are distinct α-sequences. The formal definition of $\mathrm{J}_n\alpha$ is determined, for each specific value of n, by the following recursive scheme:

$$\mathrm{J}_1\mathfrak{x} \quad \text{for} \quad \mathfrak{x},$$

$$\mathrm{J}_{k+1}\mathfrak{x} \quad \text{for} \quad (((\mathrm{J}_k\mathfrak{x} \uparrow \mathfrak{x}) \cap \mathrm{Cnv}\mathfrak{x}(\mathrm{J}_k\mathfrak{x} \uparrow \mathfrak{x})) \upharpoonright -\mathrm{I}\mathfrak{x}).$$

That this definitional scheme conforms to the given verbal description of $\mathrm{J}_n\alpha$ is seen as follows. Suppose the conformity for a given value $k(>0)$ of n. Then $\mathrm{J}_k\alpha$ is exhibited by just those mk-ads $X_1 \cdots X_k$ such that the X_i are distinct α-sequences. Consequently $(\mathrm{J}_k\alpha \uparrow \alpha)$ is exhibited by just those $(mk+m)$-ads $X_1 \cdots X_{k+1}$ such that the X_i are α-sequences all of which, with the possible exception of X_{k+1}, are distinct. By (32), then, $\mathrm{Cnv}\alpha(\mathrm{J}_k\alpha \uparrow \alpha)$ is exhibited by the results of transferring α-sequences from initial to terminal position in such $(mk+m)$-ads. These results are the $(mk+m)$-ads $X_2 \cdots X_{k+1}X_1$ such that the X_i are α-sequences all but X_{k+1} of which are distinct; renumbering, they are the $(mk+m)$-ads $X_1 \cdots X_{k+1}$ such that the X_i are α-sequences all but X_k of which are distinct. If θ be the compound $((\mathrm{J}_k\alpha \uparrow \alpha) \cap \mathrm{Cnv}\alpha(\mathrm{J}_k\alpha \uparrow \alpha))$, then, θ is exhibited by just those $(mk+m)$-ads $X_1 \cdots X_{k+1}$ such that the X_i are α-sequences which satisfy both conditions; which are mutually distinct when X_{k+1} is excepted, and are mutually distinct also when X_k is excepted; which present no identities, therefore, except possibly identity of X_k with X_{k+1}. Now, by (22), $(\theta \upharpoonright -\mathrm{I}\alpha)$ is exhibited by just those θ-sequences which end in a $-\mathrm{I}\alpha$-sequence, i.e. in a $2m$-ad whose halves are not the same α-sequence. Hence the $(\theta \upharpoonright -\mathrm{I}\alpha)$-sequences are just those $(mk+m)$-ads $X_1 \cdots X_{k+1}$ such that the X_i are α-sequences among which there is no identity, not even between X_k and X_{k+1}. Thus $\mathrm{J}_{k+1}\alpha$, defined as it is by the above recursion principle as $(\theta \upharpoonright -\mathrm{I}\alpha)$, conforms to the verbal description. The conformity for $k+1$ is thus proved, on the hypothesis of conformity for k. But the definition of $\mathrm{J}_1\alpha$ as α provides conformity for 1. The conformity for all n (>0) is thus established inductively.

Since $\mathrm{J}_n\alpha$ will be null if and only if there there are fewer than n α-sequences, the rest is obvious:

$$n\mathfrak{x} \quad \text{for} \quad (\exists \mathrm{J}_n\mathfrak{x} \cap 0\mathrm{J}_{n+1}\mathfrak{x}).$$

Wherever $n>0$, $n\alpha$ so defined is T or F according as α is or is not exhibited by just n sequences.

10. **Compass of the calculus.** The compass of the conceptual calculus will be indicated to the extent of establishing a sufficient condition for translatability into that calculus from the preliminary language of §2. But first the preliminary language must be formulated more rigorously. There are six respects in which it can be simplified without detriment to its power of expression. (a) We may restrict '=' to sequential application, since identity of concepts can be paraphrased through (1). (b) Since the choice of letters for the bound variables of quantification and abstraction is arbitrary, we may require that no letter used as a bound variable of given scope occur also in the same formula outside that scope. This enables us, in proofs of translatability, to relax the vigilance which avoidance of vicious confusion between bound and free variables otherwise re-

quires. (c) We may drop the notation of universal quantification, since $(\mathfrak{x})\mathfrak{p}$ can be paraphrased as $\sim(\exists\mathfrak{x})\sim\mathfrak{p}$. (d) We may confine abstraction to the simple case $\mathfrak{x}ʒ\mathfrak{p}$ wherein $\mathfrak{x}$ is a single VS (sequential variable, italic capital), since $\mathfrak{y}ʒ\mathfrak{q}$ can be paraphrased as $\mathfrak{x}ʒ(\exists\mathfrak{y}_1)\cdots(\exists\mathfrak{y}_n)((\mathfrak{x}=\mathfrak{y}).\mathfrak{q})$ where the $\mathfrak{y}_i$ are all the distinct VS of $\mathfrak{y}$ and $\mathfrak{x}$ is a VS not occurring in $\mathfrak{y}ʒ\mathfrak{q}$. (e) Where $\mathfrak{x}$ is a VS, $(\mathfrak{y}\epsilon\mathfrak{x}ʒ\mathfrak{z})$ can be paraphrased in each case as that expression $\mathfrak{z}'$ which is obtained from $\mathfrak{z}$ by putting $\mathfrak{y}$ for $\mathfrak{x}$ throughout. Since the preceding simplification permits the form of expression $(\mathfrak{y}\epsilon\mathfrak{x}ʒ\mathfrak{z})$ only where $\mathfrak{x}$ is a VS, we may therefore drop that form of expression entirely and confine predication to the simple case $(\mathfrak{x}\epsilon\mathfrak{c})$ where $\mathfrak{c}$ is a VC (conceptual variable, small Greek letter). (f) Of the truth-function signs we need keep only '$|$', in terms of which all truth-functions are constructible in familiar fashion. The preliminary language, simplified in these respects and subjected to a regimentation of parentheses, will be called L.

The simplifications (b)–(d) enable us to describe a *bound* VS of $\mathfrak{y}$ simply as any VS $\mathfrak{x}$ such that $(\exists\mathfrak{x})$ or $\mathfrak{x}ʒ$ occurs in $\mathfrak{y}$. A *free* VS of $\mathfrak{y}$ is a VS which occurs in $\mathfrak{y}$ but is not a bound VS of $\mathfrak{y}$. Now the *propositional expressions* (LP) of L are generated by the following recursive conditions, wherein *sequential expressions* (LS) are to be understood as VS and rows of VS:

(i) If $\mathfrak{x}$ and $\mathfrak{y}$ are LS and $\mathfrak{z}$ is a VC, $(\mathfrak{x}=\mathfrak{y})$ and $(\mathfrak{x}\epsilon\mathfrak{z})$ are LP.

(ii) If $\mathfrak{p}$ and $\mathfrak{q}$ are LP neither of which contains any bound VS of the other, $(\mathfrak{p}|\mathfrak{q})$ is an LP.

(iii) If $\mathfrak{p}$ is an LP and $\mathfrak{x}$ a free VS of $\mathfrak{p}$, $(\exists\mathfrak{x})\mathfrak{p}$ is an LP.

(Note that (ii)–(iii) are so framed as to convey the simplification (b).) The *conceptual expressions* (LC) of L comprise the VC and all expressions $\mathfrak{x}ʒ\mathfrak{p}$ such that $\mathfrak{p}$ is an LP and $\mathfrak{x}$ is a free VS of $\mathfrak{p}$.

A VS $\mathfrak{x}$ will be said to be a *fixed* VS of $\mathfrak{z}$ if the conventions of length and degree (§2) identify the length of sequences represented by $\mathfrak{x}$ (in the context $\mathfrak{z}$) with the degree of concepts represented by one of the VC of $\mathfrak{z}$, or with a quantity generable from such degrees by addition or subtraction or both. Thus an expression whose VS are fixed is free from the ambiguities arising from indeterminacies of "proper length" (§2). When a VS $\mathfrak{x}$ of $\mathfrak{z}$ is not a fixed VS of $\mathfrak{z}$, it can be fixed by inserting some tautology $\mathfrak{x}_\gamma$ after the manner of §5.

The theorem of translatability to be established is this: *Every* LP *or* LC *having no free and no unfixed* VS *is translatable into a* CC. (Cf. §3.) This is an interpretational thesis, and will be undertaken accordingly; under the intended and explained interpretations of the signs of L and of the conceptual calculus it is to be proved that, where $\mathfrak{a}$ is any one of the LP and LC in question, there in a CC $\mathfrak{c}$ which represents the same concept as $\mathfrak{a}$ for all values of the VC which are significant for $\mathfrak{a}$. (It may happen, of course, that $\mathfrak{c}$ is related to $\mathfrak{a}$ as the respective left sides of (3)–(5), (11)–(13), (15)–(19), and (22)–(29) are related to the right, i.e., that certain further values of the VC make sense of $\mathfrak{c}$ but nonsense of $\mathfrak{a}$: for a CC has meaning regardless of the degrees assigned to its VC, while this is not necessarily true of an LP or LC.) The proof occupies the next section.

11. **Proof of Translatability.**[18] Throughout the section, let $\mathfrak{a}$ be an expression

[18] I am indebted to Rosser for suggestions leading to a reduction in the length of this proof; also for the correction of a theoretical error and several mechanical ones elsewhere in the paper.

with no unfixed VS. By a *measure* of $\mathfrak{x}$ with respect to $\mathfrak{a}$ will be meant a CC $\mathfrak{y}$ such that the conventions of length and degree compel $\mathfrak{x}$ and $\mathfrak{y}$ to represent, for purposes of $\mathfrak{a}$, sequence and concept of like length and degree.

LEMMA I. *Every* LS *which occurs in* $\mathfrak{a}$, *or whose constituent* VS *occur in* $\mathfrak{a}$, *has a measure with respect to* $\mathfrak{a}$.

Proof. Where d is any one of a set of quantities, or any quantity generable from the set by addition, subtraction, or both, d can be rendered as a difference of two sums of quantities of the set. (If subtraction or addition is lacking, a quantity is added and subtracted; if subtraction occurs several times, subtrahends are collected.) From this and the definition of fixity (§10) it follows that if $\mathfrak{x}$ is a fixed VS of $\mathfrak{a}$ the conventions of length and degree call upon $\mathfrak{x}$ to represent sequences of some length $\sum_{i=1}^{h} d_i - \sum_{i=h+1}^{k} d_i$ where $1 < h < k-1$ and d_i $(1 \leqq i \leqq k)$ is taken as the degree of concepts represented by a VC $\mathfrak{y}_i$ of $\mathfrak{a}$. Since negative sequential length is not countenanced, any significant choice of degrees d_i for the several $\mathfrak{y}_i$ will be such that $\sum_{i=1}^{h} d_i \geqq \sum_{i=h+1}^{k} d_i$. But under the latter circumstance $\sum_{i=1}^{h} d_i - \sum_{i=h+1}^{k} d_i$ is the degree of concepts represented by the CC $((\mathfrak{y}_i \uparrow \cdots y_h) * (y_{h+1} \uparrow \cdots \mathfrak{y}_k))$ (with internal parentheses *ad lib.*). Hence this CC is a measure of $\mathfrak{x}$ with respect to $\mathfrak{a}$. Since every VS of $\mathfrak{a}$ is a fixed VS of $\mathfrak{a}$, every VS of $\mathfrak{a}$ has such a measure with respect to $\mathfrak{a}$. Now if the $\mathfrak{m}_j$ are measures of the respective VS $\mathfrak{x}_j$ with respect to $\mathfrak{a}$, $(\mathfrak{m}_1 \uparrow \cdots \mathfrak{m}_n)$ with internal parentheses *ad lib.* is a measure of the LS $\mathfrak{x}_1 \cdots \mathfrak{x}_n$ with respect to $\mathfrak{a}$; thus every LS composed of VS of $\mathfrak{a}$ has a measure with respect to $\mathfrak{a}$.

$\mathfrak{x}$ will be said to be *replaceable* in $\mathfrak{a}$ by $\mathfrak{y}$ if the replacement of $\mathfrak{x}$ in $\mathfrak{a}$ by $\mathfrak{y}$ does not change the meaning of $\mathfrak{a}$, i.e. forms a legitimate step of translation of $\mathfrak{a}$. (Every expression is of course replaceable in $\mathfrak{a}$ by itself, and expressions not occurring in $\mathfrak{a}$ are replaceable in $\mathfrak{a}$, vacuously, by anything.)

LEMMA II. *If* $\mathfrak{y}$ *is an* LS *whose* VS *occur in* $\mathfrak{p}$, $\mathfrak{u}$ *is an* LS *to which* $\mathfrak{y}$ *reduces upon deletion of some one occurrence of a* VS $\mathfrak{x}$, *and there is a* CC $\mathfrak{c}$ *such that* $\mathfrak{p}$ *is replaceable in* $\mathfrak{a}$ *by* $(\mathfrak{y}\epsilon\mathfrak{c})$, *then there is a* CC $\mathfrak{c}'$ *such that* $\mathfrak{p}$ *is replaceable in* $\mathfrak{a}$ *by* $(\mathfrak{u}\mathfrak{x}\epsilon\mathfrak{c}')$.

Proof. Case 1: $\mathfrak{y}$ is $\mathfrak{u}\mathfrak{x}$. Then Lemma II is verified trivially by taking $\mathfrak{c}'$ as $\mathfrak{c}$.

Case 2: $\mathfrak{y}$ is $\mathfrak{x}\mathfrak{u}$. If $\mathfrak{p}$ occurs in $\mathfrak{a}$ (and otherwise $\mathfrak{p}$ is replaceable indiscriminately in $\mathfrak{a}$, so that there is nothing to prove), then $\mathfrak{x}$ occurs in $\mathfrak{a}$ and consequently, by Lemma I, has a measure $\mathfrak{k}$ with respect to $\mathfrak{a}$. But, by (23), $(\mathfrak{x}\mathfrak{u}\epsilon\mathfrak{c})$ is interchangeable with $(\mathfrak{u}\mathfrak{x}\epsilon\mathrm{Cnv}\{\mathfrak{k}\}\mathfrak{c})$ wherever $\mathfrak{x}$ and $\mathfrak{k}$ represent sequence and concept of like length and degree. Therefore $\mathfrak{p}$, being replaceable in $\mathfrak{a}$ by $(\mathfrak{y}\epsilon\mathfrak{c})$, i.e. by $(\mathfrak{x}\mathfrak{u}\epsilon\mathfrak{c})$, is replaceable in $\mathfrak{a}$ by $(\mathfrak{u}\mathfrak{x}\epsilon\mathfrak{c}')$ where $\mathfrak{c}'$ is the CC which the formal definitions abbreviate as $\mathrm{Cnv}\{\mathfrak{k}\}\mathfrak{c}$.

Case 3: $\mathfrak{y}$ is neither $\mathfrak{u}\mathfrak{x}$ nor $\mathfrak{x}\mathfrak{u}$, and consequently has the form $\mathfrak{v}\mathfrak{x}\mathfrak{w}$ where $\mathfrak{v}\mathfrak{w}$ is $\mathfrak{u}$. If $\mathfrak{p}$ occurs in $\mathfrak{a}$, so that the VS of $\mathfrak{x}\mathfrak{w}$ occur in $\mathfrak{a}$, then, by Lemma I, $\mathfrak{x}\mathfrak{w}$ and $\mathfrak{w}$ have measures $\mathfrak{m}$ and $\mathfrak{n}$ with respect to $\mathfrak{a}$. But, by (27), $(\mathfrak{v}\mathfrak{x}\mathfrak{w}\epsilon\mathfrak{c})$ is interchangeable with $(\mathfrak{v}\mathfrak{w}\mathfrak{x}\epsilon\mathrm{S}\{\mathfrak{m}\}\{\mathfrak{n}\}\mathfrak{c})$, i.e. with $(\mathfrak{u}\mathfrak{x}\epsilon\mathrm{S}\{\mathfrak{m}\}\{\mathfrak{n}\}\mathfrak{c})$, wherever $\mathfrak{x}\mathfrak{w}$ and $\mathfrak{w}$ denote sequences of lengths equal to the respective degrees of the concepts denoted by $\mathfrak{m}$

and $\mathfrak{n}$. Hence $\mathfrak{p}$, being replaceable in $\mathfrak{a}$ by $(\mathfrak{v}\mathfrak{x}\mathfrak{w}\epsilon\mathfrak{c})$, is replaceable in $\mathfrak{a}$ by $(\mathfrak{u}\mathfrak{x}\epsilon\mathfrak{c}')$ where $\mathfrak{c}'$ is the CC which the definitions abbreviate as $\mathrm{S}\{\mathfrak{m}\}\{\mathfrak{n}\}\mathfrak{c}$.

LEMMA III. *If* $\mathfrak{y}$ *is an* LS *whose* VS *occur in* $\mathfrak{p}$, $\mathfrak{u}$ *is an* LS *to which* $\mathfrak{y}$ *is reducible by deletion of* n (>0) *occurrences of a* VS $\mathfrak{x}$, *and there is a* CC $\mathfrak{c}$ *such that* $\mathfrak{p}$ *is replaceable in* $\mathfrak{a}$ *by* $(\mathfrak{y}\epsilon\mathfrak{c})$, *then there is a* CC $\mathfrak{c}'$ *such that* $\mathfrak{p}$ *is replaceable in* $\mathfrak{a}$ *by* $(\mathfrak{u}\mathfrak{x}\epsilon\mathfrak{c}')$.

Proof. Let $\mathfrak{y}$ be reducible to $\mathfrak{v}$ by deletion of $k+1$ occurrences of $\mathfrak{x}$, and to $\mathfrak{u}$ by deletion of just k (>0) of those occurrences. Now let us assume Lemma III where $n=k$. There is then a CC $\mathfrak{c}'$ such that $\mathfrak{p}$ is replaceable in $\mathfrak{a}$ by $(\mathfrak{u}\mathfrak{x}\epsilon\mathfrak{c}')$. Since moreover deletion of a certain single occurrence of $\mathfrak{x}$ reduces $\mathfrak{u}$ to $\mathfrak{v}$ and consequently $\mathfrak{u}\mathfrak{x}$ to $\mathfrak{v}\mathfrak{x}$, it follows from Lemma II (with '$\mathfrak{u}\mathfrak{x}$', '$\mathfrak{v}\mathfrak{x}$', '$\mathfrak{c}'$', and '$\mathfrak{c}''$' put for '$\mathfrak{y}$', '$\mathfrak{u}$', '$\mathfrak{c}$', and '$\mathfrak{c}'$' therein) that there is a CC $\mathfrak{c}''$ such that $\mathfrak{p}$ is replaceable in $\mathfrak{a}$ by $(\mathfrak{v}\mathfrak{x}\mathfrak{x}\epsilon\mathfrak{c}'')$. Now if $\mathfrak{p}$ occurs in $\mathfrak{a}$, $\mathfrak{x}$ has, by Lemma I, a measure $\mathfrak{k}$ with respect to $\mathfrak{a}$. But (26) shows $(\mathfrak{v}\mathfrak{x}\mathfrak{x}\epsilon\mathfrak{c}'')$ to be interchangeable with $(\mathfrak{v}\mathfrak{x}\epsilon\mathrm{W}\{\mathfrak{k}\}\mathfrak{c}'')$ wherever $\mathfrak{x}$ and $\mathfrak{k}$ represent sequence and concept of like length and degree. Thus $\mathfrak{p}$ replaceable in $\mathfrak{a}$ by $(\mathfrak{v}\mathfrak{x}\epsilon\mathfrak{c}''')$ where $\mathfrak{c}'''$ is the CC which the definitions abbreviate as $\mathrm{W}\{\mathfrak{k}\}\mathfrak{c}''$. Lemma III (with '$\mathfrak{u}$' and '$\mathfrak{c}'$' therein rewritten as '$\mathfrak{v}$' and '$\mathfrak{c}'''$') is thus shown to hold where $n=k+1$, on the assumption that it holds where $n=k$. But it holds where $n=1$, since in this case it reduces to Lemma II. Therefore, by induction, Lemma III holds wherever $n>0$.

LEMMA IV. *If* $\mathfrak{x}$ *is a* VS *of* $\mathfrak{p}$, $\mathfrak{y}$ *is an* LS *whose sole* VS *(perhaps repeated) is* $\mathfrak{x}$, *and there is a* CC $\mathfrak{c}$ *such that* $\mathfrak{p}$ *is replaceable in* $\mathfrak{a}$ *by* $(\mathfrak{y}\epsilon\mathfrak{c})$, *then there is a* CC $\mathfrak{c}''$ *such that* $\mathfrak{p}$ *is replaceable in* $\mathfrak{a}$ *by* $(\mathfrak{x}\epsilon\mathfrak{c}'')$.

Proof. Case 1: $\mathfrak{y}$ is $\mathfrak{x}$. Then Lemma IV is verified trivially by taking $\mathfrak{c}''$ as $\mathfrak{c}$.

Case 2: $\mathfrak{y}$ comprises $n+1$ (>1) occurrences of $\mathfrak{x}$. Then deletion of n occurrences of $\mathfrak{x}$ from $\mathfrak{y}$ leaves $\mathfrak{x}$. Consequently, by Lemma III, there is a CC $\mathfrak{c}'$ such that $\mathfrak{p}$ is replaceable in $\mathfrak{a}$ by $(\mathfrak{x}\mathfrak{x}\epsilon\mathfrak{c}')$. Now if $\mathfrak{p}$ occurs in $\mathfrak{a}$, $\mathfrak{x}$ has, by Lemma I, a measure $\mathfrak{k}$ with respect to $\mathfrak{a}$. Moreover, $(\mathfrak{x}\mathfrak{x}\epsilon\mathfrak{c}')$ demands that $\mathfrak{x}\mathfrak{x}$ and $\mathfrak{c}'$ represent sequence and concept of like length and degree. But where V and VV are of lengths equal to the respective degrees of γ and β, (26) shows that $V\epsilon\mathrm{W}\{\gamma\}\beta$ if and only if $VV\epsilon\beta$; 'X' drops out of (26) as representative of the medad. Thus $\mathfrak{p}$, being replaceable in $\mathfrak{a}$ by $(\mathfrak{x}\mathfrak{x}\epsilon\mathfrak{c}')$, is replaceable in $\mathfrak{a}$ by $(\mathfrak{x}\epsilon\mathfrak{c}'')$ where $\mathfrak{c}''$ is the CC which the definitions abbreviate as $\mathrm{W}\{\mathfrak{k}\}\mathfrak{c}'$.

An LP $\mathfrak{p}$ will be said to have the property Φ under the following conditions: if $\mathfrak{p}$ has no free VS, $\mathfrak{p}$ is replaceable in $\mathfrak{a}$ by a CC; if $\mathfrak{p}$ has free VS, $\mathfrak{p}$ is replaceable in $\mathfrak{a}$ by $(\mathfrak{x}\epsilon\mathfrak{c})$ where $\mathfrak{c}$ is a CC and $\mathfrak{x}$ an LS whose VS are all and only the free VS of $\mathfrak{p}$.

LEMMA V. *If* $\mathfrak{x}$ *and* $\mathfrak{y}$ *are* LS *and* $\mathfrak{z}$ *is a* VC, $(\mathfrak{x}=\mathfrak{y})$ *and* $(\mathfrak{x}\epsilon\mathfrak{z})$ *have* Φ.

Proof. If $(\mathfrak{x}=\mathfrak{y})$ occurs in $\mathfrak{a}$, then, by Lemma I, $\mathfrak{x}$ has a measure $\mathfrak{m}$ with respect to $\mathfrak{a}$. (10) then shows $(\mathfrak{x}=\mathfrak{y})$ to be replaceable in $\mathfrak{a}$ by $(\mathfrak{x}\mathfrak{y}\epsilon\mathfrak{c})$ where $\mathfrak{c}$ is the CC which the definitions abbreviate as $\mathrm{I}\{\mathfrak{m}\}$. Since the VS of the LS $\mathfrak{x}\mathfrak{y}$ are just the free VS of $(\mathfrak{x}=\mathfrak{y})$, it follows that $(\mathfrak{x}=\mathfrak{y})$ has Φ. Since the VC $\mathfrak{z}$ is a CC, $(\mathfrak{x}\epsilon\mathfrak{z})$ has Φ as a trivial consequence of self-replaceability.

LEMMA VI. *If* $\mathfrak{p}$ *and* $\mathfrak{q}$ *have* Φ, $(\mathfrak{p}|\mathfrak{q})$ *has* Φ.

Proof. Case 1: $(\mathfrak{p}|\mathfrak{q})$ has no free VS. Then $\mathfrak{p}$ and $\mathfrak{q}$, having Φ and no free VS, are replaceable in $\mathfrak{a}$ by CC, say $\mathfrak{c}$ and $\mathfrak{c}'$, and $(\mathfrak{p}|\mathfrak{q})$ is consequently replaceable in $\mathfrak{a}$ by $(\mathfrak{c}|\mathfrak{c}')$. But it was seen in §3 that the disjunction of propositions α and β is $(\alpha * \beta)$; hence $(\mathfrak{c}|\mathfrak{c}')$ is replaceable by $(\mathfrak{c} * \mathfrak{c}')$. $(\mathfrak{p}|\mathfrak{q})$ is thus replaceable in $\mathfrak{a}$ by a CC $(\mathfrak{c} * \mathfrak{c}')$, and therefore has Φ.

Case 2: $\mathfrak{p}$ has free VS, $\mathfrak{q}$ none. Then $\mathfrak{p}$ and $\mathfrak{q}$, having Φ, are replaceable in $\mathfrak{a}$ respectively by $(\mathfrak{x}\epsilon\mathfrak{c})$ and $\mathfrak{c}'$, and $(\mathfrak{p}|\mathfrak{q})$ is consequently replaceable in $\mathfrak{a}$ by $((\mathfrak{x}\epsilon\mathfrak{c})|\mathfrak{c}')$, where $\mathfrak{c}$ and $\mathfrak{c}'$ are CC and $\mathfrak{x}$ is an LS whose VS comprise all and only the free VS of $\mathfrak{p}$, therefore all and only those of $(\mathfrak{p}|\mathfrak{q})$. Now wherever β is a proposition, so that 'Y' drops out of (3), (3) shows $(\alpha * \beta)$ to be the concept exhibited by just those sequences X such that $\sim((X\epsilon\alpha).\beta)$, i.e. such that $((X\epsilon\alpha)|\beta)$. Hence $((\mathfrak{x}\epsilon\mathfrak{c})|\mathfrak{c}')$ is interchangeable with $(\mathfrak{x}\epsilon(\mathfrak{c} * \mathfrak{c}'))$. $(\mathfrak{p}|\mathfrak{q})$ is thus replaceable in $\mathfrak{a}$ by $(\mathfrak{x}\epsilon(\mathfrak{c} * \mathfrak{c}'))$, and consequently has Φ.

Case 3: $\mathfrak{q}$ has free VS, $\mathfrak{p}$ none. Since $(\mathfrak{p}|\mathfrak{q})$ is interchangeable with $(\mathfrak{q}|\mathfrak{p})$, this case reduces to Case 2.

Case 4: $\mathfrak{p}$ and $\mathfrak{q}$ both have free VS. $(\mathfrak{p}|\mathfrak{q})$ is then replaceable in $\mathfrak{a}$ by $((\mathfrak{x}\epsilon\mathfrak{c})|(\mathfrak{y}\epsilon\mathfrak{c}'))$ where $\mathfrak{c}$ and $\mathfrak{c}'$ are CC and $\mathfrak{x}$ and $\mathfrak{y}$ are LS containing respectively just the free VS of $\mathfrak{p}$ and just those of $\mathfrak{q}$. Since disjunction is the denial of conjunction, (7) and (2) show $((\mathfrak{x}\epsilon\mathfrak{c})|(\mathfrak{y}\epsilon\mathfrak{c}'))$ to be replaceable in turn by $(\mathfrak{x}\mathfrak{y}\epsilon\mathfrak{c}'')$ where $\mathfrak{c}''$ is the CC which the definitions abbreviate as $-(\mathfrak{c}\uparrow\mathfrak{c}')$. Therefore $(\mathfrak{p}|\mathfrak{q})$ has Φ, since the VS of $\mathfrak{x}\mathfrak{y}$ are all and only the free VS of $(\mathfrak{p}|\mathfrak{q})$.

LEMMA VII. *If* $\mathfrak{p}$ *has* Φ *and* $\mathfrak{x}$ *is a free* VS *of* $\mathfrak{p}$, $(\exists\mathfrak{x})\mathfrak{p}$ *has* Φ.

Proof. Since $\mathfrak{p}$ has Φ and has at least one free VS, $\mathfrak{p}$ is replaceable in $\mathfrak{a}$ by $(\mathfrak{y}\epsilon\mathfrak{c})$ where $\mathfrak{c}$ is a CC and $\mathfrak{y}$ an LS whose VS are all and only the free VS of $\mathfrak{p}$.

Case 1: $(\exists\mathfrak{x})\mathfrak{p}$ has no free VS. Then $\mathfrak{x}$ is the only free VS of $\mathfrak{p}$, for any further one would be a free VS also of $(\exists\mathfrak{x})\mathfrak{p}$. $\mathfrak{x}$ is thus the only VS of $\mathfrak{y}$. Hence, by Lemma IV, there is a CC $\mathfrak{c}''$ such that $\mathfrak{p}$ is replaceable in $\mathfrak{a}$ by $(\mathfrak{x}\epsilon\mathfrak{c}'')$, i.e. such that $(\exists\mathfrak{x})\mathfrak{p}$ is replaceable in $\mathfrak{a}$ by $(\exists\mathfrak{x})(\mathfrak{x}\epsilon\mathfrak{c}'')$. But $(\exists\mathfrak{x})(\mathfrak{x}\epsilon\mathfrak{c}'')$ is replaceable in turn by that CC which the definitions abbreviate as $\exists\mathfrak{c}''$. Therefore $(\exists\mathfrak{x})\mathfrak{p}$ has Φ.

Case 2: $(\exists\mathfrak{x})\mathfrak{p}$ has free VS. Then $\mathfrak{y}$ contains those VS in addition to $\mathfrak{x}$. By Lemma III, there is a CC $\mathfrak{c}'$ such that $\mathfrak{p}$ is replaceable in $\mathfrak{a}$ by $(\mathfrak{u}\mathfrak{x}\epsilon\mathfrak{c}')$ where $\mathfrak{u}$ is the result of deleting all occurrences of $\mathfrak{x}$ from $\mathfrak{y}$. $(\exists\mathfrak{x})\mathfrak{p}$, then, is replaceable in $\mathfrak{a}$ by $(\exists\mathfrak{x})(\mathfrak{u}\mathfrak{x}\epsilon\mathfrak{c}')$. Now if $(\exists\mathfrak{x})\mathfrak{p}$ occurs in $\mathfrak{a}$, $\mathfrak{x}$ has, by Lemma I, a measure $\mathfrak{k}$ with respect to $\mathfrak{a}$. But (12) shows $(\exists\mathfrak{x})(\mathfrak{u}\mathfrak{x}\epsilon\mathfrak{c}')$ to be interchangeable with $(\mathfrak{u}\epsilon(\mathfrak{c}'''\{\mathfrak{k}\}))$ wherever $\mathfrak{x}$ and $\mathfrak{k}$ represent sequence and concept of like length and degree. Therefore $(\exists\mathfrak{x})\mathfrak{p}$ is replaceable in $\mathfrak{a}$ by $(\mathfrak{u}\epsilon\mathfrak{c}'')$ where $\mathfrak{c}''$ is the CC which the definitions abbreviate as $(\mathfrak{c}'''\{\mathfrak{k}\})$. Moreover, the VS of $\mathfrak{u}$ are all and only the free VS of $(\exists\mathfrak{x})\mathfrak{p}$: for they are all the free VS of $\mathfrak{p}$ except $\mathfrak{x}$. Therefore $(\exists\mathfrak{x})\mathfrak{p}$ has Φ.

LEMMA VIII. *Every* LP *has* Φ.

Proof. By (i)–(iii) of §10, every LP must either be of one of the forms covered by Lemma V, or be built up ultimately of such LP by one or more applications of one or both of the methods dealt with in Lemmas VI and VII. Lemma VIII therefore follows from Lemmas V–VII by induction.

LEMMA IX. *If* $\mathfrak{a}$ *is an* LP *with no free* VS, $\mathfrak{a}$ *is translatable into a* CC.

Proof. By Lemma VIII, $\mathfrak{a}$ has Φ; hence, having no free VS, $\mathfrak{a}$ is replaceable in $\mathfrak{a}$ by a CC. But replacement of $\mathfrak{a}$ in *itself* by a CC is simply translation of $\mathfrak{a}$ into that CC.

LEMMA X. *If* $\mathfrak{a}$ *is an* LC *with no free* VS, $\mathfrak{a}$ *is translatable into a* CC.

Proof. By §10, every LC is either a VC, and hence already a CC, or else has the form $\mathfrak{x}$ʒ$\mathfrak{p}$ where $\mathfrak{x}$ is a free VS of the LP $\mathfrak{p}$. If $\mathfrak{a}$ is $\mathfrak{x}$ʒ$\mathfrak{p}$, then, since $\mathfrak{a}$ contains no free VS, $\mathfrak{x}$ is the only free VS of $\mathfrak{p}$. But, by Lemma VIII, $\mathfrak{p}$ has Φ, and is hence replaceable in $\mathfrak{a}$ by $(\mathfrak{y}\epsilon\mathfrak{c})$ where $\mathfrak{c}$ is a CC and the VS of $\mathfrak{y}$ comprise just the free VS of $\mathfrak{p}$, therefore just $\mathfrak{x}$. By Lemma IV, then, $\mathfrak{p}$ is replaceable in $\mathfrak{a}$ by $(\mathfrak{x}\epsilon\mathfrak{c}')$ for some CC $\mathfrak{c}'$. But replacement of $\mathfrak{p}$ in $\mathfrak{a}$ by $(\mathfrak{x}\epsilon\mathfrak{c}')$ turns $\mathfrak{a}$ into $\mathfrak{x}$ʒ$(\mathfrak{x}\epsilon\mathfrak{c}')$, which is translatable as $\mathfrak{c}'$ itself.

$\mathfrak{a}$ was taken originally as any expression having no unfixed VS. Hence Lemmas IX and X show that every LP or LC having no free and no unfixed VS is translatable into a CC, *q.e.d.*

12. **Agenda.** The next desideratum is a formal set of postulates, expressed solely in terms of the primitives of the calculus and hence comprised among the CC, and a set of rules of inference generating further CC from the postulates by way of theorems. Being CC, the postulates and theorems will denote concepts; specifically they will denote medadic concepts or propositions, more specifically the true one T. What is wanted is thus a minimum array of CC denoting T for all values of their variables, and a minimum set of rules whereby any further CC can be generated from the former if and only if it denotes T for all values of its variables. The task is one of systematizing existing doctrine, for existing logistic determines, relatively to the presented interpretation of our primitives, which CC denote T.

Perhaps two rules of inference will suffice: the familiar rule of substitution for variables, and some such second rule as this: *Given* $\mathfrak{x}$ *and* $((\mathfrak{y} * \mathfrak{x}) * \mathfrak{x})$ *as theorems or postulates, take* $\mathfrak{y}$ *as a theorem.* That this rule is admissible, in the sense of leading from designations of T only to designations of T, is seen as follows: Where $\mathfrak{x}$ denotes T, D5 shows $((\mathfrak{y} * \mathfrak{x}) * \mathfrak{x})$ to be synonymous with $--\mathfrak{y}$ and hence with $\mathfrak{y}$; where $\mathfrak{x}$ and $((\mathfrak{y} * \mathfrak{x}) * \mathfrak{x})$ both denote T, therefore, $\mathfrak{y}$ must denote T. The isolation of a few CC denoting T, from which as postulates every CC denoting T is generable as a theorem by some such pair of rules of inference, remains to be accomplished.

A second *agendum* is the erection upon the conceptual calculus of an adequate superstructure treating concepts of higher types and involving bound VC of all types. It is hoped that this can be accomplished merely by a set of notational conventions relative to the conceptual calculus. The general nature of the approach will now be sketched.

In a notation due to Carnap[19] $(t_1 : \cdots t_n)$ is the type of concepts significantly predicable of n-ads whose successive places are occupied by entities of the respective types t_i. Thus, if we take individuals as of type 0, the concepts which

[19] *Logische Syntax der Sprache,* p. 76.

are the elements of the conceptual calculus are of types of the form $(0: \cdots 0)$. But the conceptual calculus departs from ordinary procedure in not segregating $()$, (0), $(0{:}0)$, $(0{:}0{:}0)$, etc., as separate types; from the present standpoint these may be designated jointly as type 1. Concepts which are attributes or relations of the concepts which are elements of the calculus, then, have types of the form $(1: \cdots 1)$. Such concepts can be introduced by the following convention:

(I) *Where the* $\mathfrak{u}_i$ *are* CC *and the* $\mathfrak{z}_i$ *are distinct* VC, *let* $((\mathfrak{u}_1, \cdots \mathfrak{u}_m)\epsilon(\mathfrak{z}_1, \cdots \mathfrak{z}_m)\ni\mathfrak{x})$ *be used as alternative notation for that expression* $\mathfrak{x}'$ *which comes of putting the successive* $\mathfrak{u}_i$ *for the respective* $\mathfrak{z}_i$ *throughout* $\mathfrak{x}$.

(I) need interest us in only those cases where $\mathfrak{x}$ and $\mathfrak{x}'$ represent propositions, rather than non-medadic concepts; the expression introduced by (I) then expresses a proposition likewise, and may be read as affirming that the m-ad of concepts represented by $(\mathfrak{u}_1, \cdots \mathfrak{u}_m)$ exhibits the m-adic concept represented by $(\mathfrak{z}_1, \cdots \mathfrak{z}_m)\ni\mathfrak{x}$. The expression $(\mathfrak{z}_1, \cdots \mathfrak{z}_m)\ni\mathfrak{x}$, which proceeds from (I) only as an incomplete symbol,[20] comes thus to be viewed as representing an m-adic concept of type $(1: \cdots 1)$: an m-adic relation, or, where $m=1$, an attribute, of concepts of type 1.[21] Now we can subject these concepts of types $(1: \cdots 1)$ to the first primitive operation of the conceptual calculus through the following convention:

(II) *Where* $\mathfrak{a}$ *is* $(\mathfrak{z}_1, \cdots \mathfrak{z}_m)\ni\mathfrak{x}$, $\mathfrak{b}$ *is* $(\mathfrak{w}_1, \cdots \mathfrak{w}_n)\ni\mathfrak{y}$, *and the* $\mathfrak{u}_i$ *are distinct* VS *not occurring in* $\mathfrak{a}$ *or* $\mathfrak{b}$, *use* $(\mathfrak{a} \uparrow \mathfrak{b})$ *for* $(\mathfrak{u}_1, \cdots \mathfrak{u}_{m+n})\ni(((\mathfrak{u}_1, \cdots \mathfrak{u}_m)\epsilon\mathfrak{a}) \uparrow ((\mathfrak{u}_{m+1}, \cdots \mathfrak{u}_{m+n})\epsilon\mathfrak{b}))$.

We are interested here, as before, only in those cases where $((\mathfrak{u}_1, \cdots \mathfrak{u}_m)\epsilon\mathfrak{a})$ and $((\mathfrak{u}_{m+1}, \cdots \mathfrak{u}_{m+n})\epsilon\mathfrak{b})$ express propositions. In such cases the arrow joining them expresses conjunction, as observed in §3, so that (II) explains $(\mathfrak{a} \uparrow \mathfrak{b})$ in a fashion precisely paralleling (2). Now if for the moment we suppose universal quantification with respect to VC already to have been introduced somehow through notational conventions, we can subject the concepts of types $(1: \cdots 1)$ to the other primitive operation through the following convention:

(III) *Where* $\mathfrak{a}$, $\mathfrak{b}$, *and the* $\mathfrak{u}_i$ *are as in* (II), *and* $\mathfrak{v}$ *is* $(((\mathfrak{u}_1, \cdots \mathfrak{u}_m)\epsilon\mathfrak{a}) * ((\mathfrak{u}_1, \cdots \mathfrak{u}_n)\epsilon\mathfrak{b}))$, *use* $(\mathfrak{a} * \mathfrak{b})$ *for* $(\mathfrak{u}_1, \cdots \mathfrak{u}_{m-n})\ni(\mathfrak{u}_{m-n+1}) \cdots (\mathfrak{u}_n)\mathfrak{v}$ *or for* $(\mathfrak{u}_1) \cdots (\mathfrak{u}_m)\mathfrak{v}$ *or for* $(\mathfrak{u}_1, \cdots \mathfrak{u}_{n-m}, \mathfrak{u}_{n+1}, \cdots \mathfrak{u}_{2n-m})\ni(\mathfrak{u}_{n-m+1}) \cdots (\mathfrak{u}_n)(\cdot\cdot (\mathfrak{v} \cap (\mathfrak{u}_1 = \mathfrak{u}_{n+1}))\cap \cdots (\mathfrak{u}_{n-m} = \mathfrak{u}_{2n-m}))$ *according as* $m >, =,$ *or* $< n$.

Again we are interested only in the cases where $((\mathfrak{u}_1, \cdots \mathfrak{u}_m)\epsilon\mathfrak{a})$ and $((\mathfrak{u}_1, \cdots, \mathfrak{u}_n)\epsilon\mathfrak{b})$ express propositions. In such cases $\mathfrak{v}$ represents their disjunction, as observed in §3, so that (III) explains $(\mathfrak{a} * \mathfrak{b})$ in a fashion precisely paralleling (3)–(5). Now all the formal definitions of the conceptual calculus, based as they are upon the two primitives, come to apply in parallel fashion to concepts of types $(1: \cdots 1)$ through the following convention:

(IV) *Where* $\mathfrak{y}$ *is a* CC *and the* $\mathfrak{w}_i$ *are distinct* VC, *let* $(\mathfrak{w}_1, \cdots \mathfrak{w}_n)\ni\mathfrak{y}$ *be subject to all conventions to which all* CC *are subject.*

Through (IV), moreover, (I)–(III) come to treat concepts of successively

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[20] In this respect the present procedure agrees with that of *PM*, q.v., pp. 66, 71–72, 81, 187–188, 200.

[21] In the more general case where $\mathfrak{x}$ may represent concepts other than propositions, $(\mathfrak{z}_1, \cdots \mathfrak{z}_m)\ni\mathfrak{x}$ answers rather to Church's $\lambda\mathfrak{z}_1 \cdots \lambda\mathfrak{z}_m . \mathfrak{x}$. Cf. his *A set of postulates for the foundation of logic*, ***Annals of mathematics***, 2 s. vol. 33 (1932), pp. 351–354.

higher types. Thus (IV) tells us that (I), enunciated as it is for all CC $\mathfrak{u}_i$, is to apply also in the following extended form:

(a) *Where the several* $\mathfrak{u}_i$ *are* CC *or expressions of the form* $(\mathfrak{w}_1, \cdots \mathfrak{w}_n)\ni\mathfrak{y}$, *and the* $\mathfrak{z}_i$ *are distinct* VC, *let* [*etc. as in* (I)].

(a) introduces expressions $(\mathfrak{z}_1, \cdots \mathfrak{z}_m)\ni\mathfrak{x}$ denoting concepts whose types are of the form $(t_1: \cdots t_m)$ where the t_i are severally 1 or of the form $(1: \cdots 1)$. Now some or all of the expressions $(\mathfrak{w}_1, \cdots \mathfrak{w}_n)\ni\mathfrak{y}$ referred to in (a) may henceforward be of this new sort; they may represent concepts, not necessarily of types $(1: \cdots 1)$, but of types $(t_1: \cdots t_n)$ where the t_i are severally 1 or of the form $(1: \cdots 1)$. Thereupon (a) comes to introduce expressions $(\mathfrak{z}_1, \cdots \mathfrak{z}_m)\ni\mathfrak{x}$ denoting concepts of types $(s_1: \cdots s_m)$ where the s_i are severally 1 or of the form $(t_1: \cdots t_n)$ where the t_i are severally 1 or of the form $(1: \cdots 1)$. Now some or all of the expressions $(\mathfrak{w}_1, \cdots \mathfrak{w}_n)\ni\mathfrak{y}$ referred to in (a) may be taken as of this newly achieved variety; and so on, *ad infinitum*. In this fashion (a), the expansion of (I) under (IV), extends its own scope to concepts of progressively higher type without limit. (II), (III), the quantification convention presupposed in (III), and the formal definitions of the conceptual calculus, all correspondingly expanded under (IV), then serve to project all notions of the conceptual calculus through all these higher types.

Under the above developments the VC cease to be confined to the representation of concepts of type 1, and take on a typical ambiguity similar to that which the class variables enjoy in *Logistic*.[22] Associated types come to depend upon context. Thus when $\mathfrak{u}_k$ is an expression $(\mathfrak{w}_1, \cdots \mathfrak{w}_n)\ni\mathfrak{y}$, denoting let us say a concept of type $(t_1: \cdots t_n)$, the bound VC $\mathfrak{z}_k$ in $((\mathfrak{u}_1, \cdots \mathfrak{u}_m)\epsilon(\mathfrak{z}_1, \cdots \mathfrak{z}_m)\ni\mathfrak{x})$ comes to stand for concepts not of type 1 but of type $(t_1: \cdots t_n)$. The whole expression $((\mathfrak{u}_1, \cdots \mathfrak{u}_m)\epsilon(\mathfrak{z}_1, \cdots \mathfrak{z}_m)\ni\mathfrak{x})$ will be meaningful, i.e. an explained transcription of a CC, only if it is so explained by (I) and the derivative (a) of (I) under (IV): hence only if $\mathfrak{x}$ is such that from substitution of $(\mathfrak{w}_1, \cdots \mathfrak{w}_n)\ni\mathfrak{y}$ for $\mathfrak{z}_k$ in $\mathfrak{x}$, and substitution similarly of the other $\mathfrak{u}_i$ for the other $\mathfrak{z}_i$, an expression results which is a CC or is explained in turn by (I)–(IV) and antecedent definitions as a transcription ultimately of a CC. Expressions which do not so reduce are meaningless, and these turn out to embrace all those expressions, e.g. expressions of the form $(\mathfrak{x}\epsilon\mathfrak{x})$, which violate the familiar theory of types.

The notation of the usual logistic is broader than the ideas which the notation is intended to express; e.g., the general notational scheme $(\mathfrak{x}\epsilon\mathfrak{y})$ includes the special case $(\mathfrak{x}\epsilon\mathfrak{x})$, whereas the logical paradoxes teach that under the given interpretation of 'ϵ' there are no propositions corresponding to the notation $(\mathfrak{x}\epsilon\mathfrak{x})$. The theory of types was for this reason appended to logistic, not as part of the formal deductive system, but as an essential metamathematical auxiliary serving to sort the expressions generable from the primitive notation and to discard (i.e. deny meaning to) waste bits such as $(\mathfrak{x}\epsilon\mathfrak{x})$. The procedure sketched above, on the other hand, avoids the need of this anomalous appendage. Expressions for whose exclusion the theory of types was formerly needed now become meaningless simply by virtue of their irreducibility, through definitions and kindred notational con-

[22] Pp. 19, 39, 85.

ventions, to CC: meaningless in the ordinary sense of being neither primitives nor constructs of primitives.

So much by way of a rough indication of the contemplated procedure. Concepts of types $(t_1: \cdots t_n)$, wherein some but not all of the t_i are 0, remain untouched by (I)–(IV); provision for them is a supplementary problem, which will not be entered upon here. Another omitted feature is the introduction of quantification with respect to VC, supposed accomplished preparatory to (III). One course to this end would be adoption of the following conventions, supplemented by the familiar definitions of other truth-functions in terms of disjunction:

(V) Where $\mathfrak{z}$ is a VC, prefix $(\mathfrak{z})$ at pleasure to theorems as an idle ornament.

(VI) Where $\mathfrak{z}$ does not occur in $\mathfrak{x}$, and $\mathfrak{y}'$ is the result of putting the VC $\mathfrak{w}$ for $\mathfrak{z}$ throughout $\mathfrak{y}$, use $(\mathfrak{x} \mid (\exists\mathfrak{w})\mathfrak{y}')$ for $(\mathfrak{z}) - (\mathfrak{x} \cap \mathfrak{y})$, $((\exists\mathfrak{w})\mathfrak{y}' \mid \mathfrak{x})$ for $(\mathfrak{z}) - (\mathfrak{y} \cap \mathfrak{x})$, $(\mathfrak{x} \mid (\mathfrak{w})\mathfrak{y}')$ for $(\exists\mathfrak{z}) - (\mathfrak{x} \cap \mathfrak{y})$, and $((\mathfrak{w})\mathfrak{y}' \mid \mathfrak{x})$ for $(\exists\mathfrak{z}) - (\mathfrak{y} \cap \mathfrak{x})$.

In recursive fashion these two conventions explain quantification of any disjunctive part of a theorem, overlaid by no matter how many layers of disjunction, provided only that the quantification is universal or particular according as that number of layers is even or odd. This limitation amounts to the proscription of existence-theorems at the higher type-levels, and thus represents a step toward intuitionism.

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